An OPRemez Approach to the Design of FIR Digital Filters*

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Abstract: This paper considers the minimax design problem of FIR digital filters and proposes a new design algorithm. The proposed algorithm is a Remez exchange based one in which the coefficients of the filter are evaluated with an orthogonalized projection algorithm. The estimated coefficients are proved to be convergent to the true values for a given set of extremal frequencies. Design examples show that the proposed OPRemez algorithm can be used to design filters with very high filter length, and that the algorithm takes fewer exchanges than the standard Remez exchange algorithm.

Keywords: FIR filters; Remez exchange; orthogonalized projection

1. Introduction

The design of FIR filters is readily amenable from a variety of directions such as exactly linear phase, always stable and finite duration of impulse response. Moreover, due to the important application of the distortion-free transmission of wave-forms in the passband, FIR filters are attractive because the linear phase or constant group delay requirement can be easily satisfied. In the past decades, there have been developed many algorithms for the design of FIR filters such as window function [1,2], Remez exchange [1,3,4] and least-squares [5,6]. Among others, Remez exchange algorithm is most popular because it gives optimal designs in the minimax sense with constant group delay. It is often used as a standard to be compared with other newly developed methods. However, the standard Remez algorithm is very complicated. It employs interpolation technique, multiple exchange technique and Fourie transformation technique to accomplish the design.

The orthogonalized projection (OP) algorithm[7] is an recursive identification algorithm for deterministic linear systems. The algorithm is actually a way of sequentially solving a set of linear equations for the unknown parameters of the system. With the orthogonalized projection algorithm, the set of linear equations can be easily solved without matrix inversion. The algorithm has good numerical performance.

This paper presents an OPRemez algorithm for the minimax design of FIR filters. It applies the orthogonalized projection algorithm to calculate the filter's coefficients directly from a supposed set of extremal frequencies and then replace the old set of extremal frequencies with a new set of frequencies where the errors are maximal. The proposed algorithm is simpler and takes fewer exchanges than the standard Remez exchange algorithm. The design examples demonstrate the effectiveness of the proposed algorithm.

2. The Design Problem and the Standard Remez Exchange Algorithm

The frequency response of an FIR filter with filter length N can be expressed as

$$H(e^{j\omega})=\sum_{n=0}^{N-1}h(n)e^{-jn\omega}\equiv Q(\omega)e^{j\varphi(\omega)},$$

where $Q(\omega)$ and $\varphi(\omega)$ are the magnitude and the phase frequency response of the filter. For simplicity, consider filters with odd N and symmetric impulse response, i.e., $h(n)=h(N-1-n)$. In this case, $\varphi(\omega)=-\frac{N-1}{2}\omega$ is linear and

$$Q(\omega)=\sum_{n=0}^{r-1}\alpha_n\cos n\omega\,, \tag{1}$$

where $r=\frac{N-1}{2}+1$, $\alpha_0=h(r-1)$, $\alpha_n=2h(r-1-n)$ ($n\neq 0$). The design of the FIR filter means choosing a set of coefficients $\{\alpha_n, n=0,1,\cdots, r-1\}$ in order that $Q(\omega)$ approximates the desired real response $D(\omega)$. In the

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Chebechev (minimax) sense, the design problem can be formulated as

$$\min_{\{\alpha_n\}}\left[\max_{\omega\in A}|E(\omega)|\right], \qquad (2)$$

where

$$E(\omega)=W(\omega)[D(\omega)-Q(\omega)]$$

is the approximation error weighted by a weight function $W(\omega)$ and A is a close subset of interval $[0,\pi]$.

The Chebechev approximation has famous properties summarized as the alternating theorem [1]. These properties can be used to find out the solution to the design problem described by equation (2). The alternating theorem is as follow.

Alternating Theorem: Let $Q(\omega)$, as in equation (1), be a linear combination of r cosine functions and $D(\omega)$ be a continuous function defined on A. $Q(\omega)$ is the unique and best weighted Chebychev approximation to $D(\omega)$ if and only if there exists a set of at least $(r+1)$ frequencies

$$0\le\omega_1<\omega_2<\cdots<\omega_r<\omega_{r+1}\le\pi \qquad (3)$$

such that

$$E(\omega_k)=-E(\omega_{k+1}) \quad (k=1,2,\cdots,r)$$

and

$$|E(\omega_k)|=\max_{\omega\in A}|E(\omega)|.$$

This set of $(r+1)$ frequencies is called a set of extremal frequencies.

According to the alternating theorem, if the extremal frequencies $\{\omega_k, k=1,\cdots,r+1\}$ are known, we can obtain a set of $(r+1)$ equations as follow,

$$W(\omega_k)[D(\omega_k)-Q(\omega_k)]=(-1)^k\delta \ \ (k=1,\cdots,r+1), \qquad (4)$$

where $\delta=\max_{\omega\in A}|E(\omega)|$ is the maximal weighted error. Using equation (1), the above set of equations can be written in a matrix form as

$$G[\alpha_0\cdots\alpha_{r-1}\ \delta]^T=[D(\omega_1)\cdots D(\omega_r)\,D(\omega_{r+1})]^T, \qquad (5)$$

where

$$G=\begin{bmatrix}1 & \cos\omega_1 & \cdots & \cos(r-1)\omega_1 & W^{-1}(\omega_1)\\ 1 & \cos\omega_2 & \cdots & \cos(r-1)\omega_2 & -W^{-1}(\omega_2)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & \cos\omega_r & \cdots & \cos(r-1)\omega_r & (-1)^{r-1}W^{-1}(\omega_r)\\ 1 & \cos\omega_{r+1} & \cdots & \cos(r-1)\omega_{r+1} & (-1)^r W^{-1}(\omega_{r+1})\end{bmatrix}.$$

From (5), the filter's coefficients $\alpha_0,\alpha_1,\cdots,\alpha_{r-1}$ and the maximum error δ can be solved.

But in fact the extremal frequencies are unknown, $\alpha_0,\alpha_1,\cdots,\alpha_{r-1}$ and δ cannot be solved directly from equation (5). J.H. McClellan, T.W. Parks and L.R. Rabiner[3,4] applied Remez exchange algorithm in the approximation theory to evaluate the set of extremal frequencies in an iterative scheme, starting from an initial set of frequencies. In addition, it is generally understood that it is very difficult and slow to solve equation (5) directly with given frequencies $\{\omega_k, k=1,\cdots,r+1\}$. L.R. Rabiner[1] proposed an algorithm (the standard Remez exchange algorithm) in which δ is first computed with an analytical formula educed from equation (5), and $\{Q(\omega_k), k=1,\cdots,r+1\}$ are then calculated from (4) and then used to determine $Q(\omega)$ for the entire frequency interval $[0,\pi]$ by Lagrange interpolation. A new set of extremal frequencies is evaluated from $Q(\omega)$ and the iterative procedure goes on. After the procedure converges, the coefficients $\alpha_0,\alpha_1,\cdots,\alpha_{r-1}$ and the maximum error δ are then calculated with the inverse Fourie transform.

In this paper, the coefficients $\alpha_0,\alpha_1,\cdots,\alpha_{r-1}$ and the maximum error δ will be calculated directly by solving equation (5) with the orthogonalized projection algorithm[7] described in next section. The extremal frequencies are also updated by Remez multiple exchange. The proposed design method is a combination of orthogonal projection and Remez exchange.

3. The orthogonalized projection algorithm

Consider a dynamic system described by a model in the following simple form:

$$y(k)=\phi(k)^T\theta, \qquad (6)$$

where $y(k)$, $\phi(k)$ and θ denote the scalar output, the data vector and the unknown parameter vector of the system at time k respectively. Assume that a sequence of observations on both $y(\bullet)$ and $\phi(\bullet)$ is available, then the unknown parameter vector can be estimated recursively by the algorithm below.

Orthogonalized Projection (OP) Algorithm

For $k=1,2,3,\cdots$, if $\phi(k)^T P(k-1)\phi(k)\neq 0$, then

$$e(k)=y(k)-\phi(k)^T\hat\theta(k-1)$$

$$\hat\theta(k)=\hat\theta(k-1)+\frac{P(k-1)\phi(k)}{\phi(k)^T P(k-1)\phi(k)}e(k), \qquad (7)$$

$$P(k)=P(k-1)-\frac{P(k-1)\phi(k)\phi(k)^T P(k-1)}{\phi(k)^T P(k-1)\phi(k)},$$

if $\phi(k)^T P(k-1)\phi(k)=0$, then

$$\hat\theta(k)=\hat\theta(k-1), \quad P(k)=P(k-1), \qquad (8)$$

where the initial estimate $\hat\theta(0)$ is given and $P(0)=I$.

Typical properties of above algorithm are listed in the lemma below.

Lemma 1 Consider the orthogonalized projection algorithm subject to the model (6), then

(i) $P(k-1)\phi(k)$ is orthogonal to $\phi(1),\cdots,\phi(k-1)$.

(ii) $\tilde{\theta}(k)^T \phi(i)=0$ for $1\le i\le k$, where

$$\tilde{\theta}(k)=\hat{\theta}(k)-\theta .$$

Proof: See reference [7].

It is seen that the vector $P(k-1)\phi(k)$ in the algorithm above is the component of $\phi(k)$ which is orthogonal to all previous $\phi(\bullet)$ vectors. The matrix $P(k-1)$ is a projection operator that ensures this property. Property (ii) means that at step *k*, the estimated parameter vector $\hat{\theta}(k)$ satisfies all previous equations of the system, i.e., $y(i)=\phi(i)^T\hat{\theta}(k)$ for $1\le i\le k$. So the algorithm is similar to a way of sequentially solving the set of linear equations for the unknown parameters of the system. The convergence of the orthogonalized projection algorithm is given in the following theorem.

Theorem 2 The orthogonalized projection algorithm subject to the model (6) will converge to θ in m steps provided that

$$\text{rank}\,[\phi(1),\phi(2),\cdots,\phi(m)]=n=\text{dimension of } \theta .$$

Proof: Immediately from part (ii) of lemma 1.

4. The OPRemez Design Algorithm for Linear Phase FIR Filters

Now define

$$y(k)=D(\omega_k), \tag{9}$$

$$\phi(k)=[1,\cos\omega_k,\cdots,\cos(r-1)\omega_k,(-1)^{k-1}W^{-1}(\omega_k)]^T, \tag{10}$$

$$\theta=[\alpha_0,\alpha_1,\cdots,\alpha_{r-1},\delta]^T, \tag{11}$$

then equation (5) can be rewritten in the form of equation (6), i.e.,

$$y(k)=\phi(k)^T\theta \text{ with } k=1,2,\cdots,r+1. \tag{12}$$

For a given set of frequencies $\{\omega_k, k=1,\cdots,r+1\}$, a set of inputs and outputs $\{\phi(k), y(k), k=1,\cdots,r+1\}$ is available. If the orthogonalized projection algorithm is applied to estimate the parameter vector θ, the estimated vector $\hat{\theta}(r+1)$ will be equal to θ. Lemma 3 and theorem 4 given below establish this result.

Lemma 3 If $(r+1)$ frequencies $\{\omega_k, k=1,\cdots,r+1\}$ satisfy equation (3) and $W(\omega_k)>0$, then

$$\text{rank}\,[\phi(1),\phi(2),\cdots,\phi(r+1)]=r+1.$$

Proof Define determinant $V(\omega_1,\cdots,\omega_{m+1})$ as

$$V(\omega_1,\cdots,\omega_{m+1})\equiv\begin{vmatrix}1 & \cos\omega_1 & \cos 2\omega_1 & \cdots & \cos m\omega_1\\ 1 & \cos\omega_2 & \cos 2\omega_2 & \cdots & \cos m\omega_2\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & \cos\omega_{m+1} & \cos 2\omega_{m+1} & \cdots & \cos m\omega_{m+1}\end{vmatrix}.$$

With the help of the properties of Vandermond determinants and the formula

$$\cos(m+1)\omega=2\cos\omega\cos m\omega-\cos(m-1)\omega,$$

it can be easily obtained that

$$V(\omega_1,\cdots,\omega_m)=\frac{V(\omega_1,\cdots,\omega_{m+1})}{2^{m-1}}\prod_{i=1}^{m}\frac{1}{(\cos\omega_{m+1}-\cos\omega_i)}.$$

Now define $X=[\phi(1),\phi(2),\cdots,\phi(r+1)]^T$. By using the cofactor expansion, it follows from above equation that

$$\det X=2^{-(r-1)}V(\omega_1,\omega_2,\cdots,\omega_{r+1})\sum_{k=1}^{r+1}(-1)^k W^{-1}(\omega_k)d_k,$$

where

$$d_k=\prod_{i=1,i\ne k}^{r+1}(\cos\omega_k-\cos\omega_i)^{-1}.$$

Because the $(r+1)$ frequencies satisfy equation (3) and $W(\omega_k)>0$, $V(\omega_1,\omega_2,\cdots,\omega_{r+1})\ne 0$ and $(-1)^K d_k * W^{-1}(\omega_k)<0$ for $k=1,2,\cdots,r+1$. It follows that $\det X\ne 0$, which implies the result of this lemma.

Theorem 4 Consider equations (9)~(12). For given frequencies as in (3) and $W(\omega_k)>0$, if the orthogonalized projection algorithm is used to estimate the parameter vector θ, then the estimated parameter vector $\hat{\theta}(k)$ will converge to θ in $(r+1)$ steps, that is,

$$\tilde{\theta}(r+1)=\hat{\theta}(r+1)-\theta=0.$$

Proof Immediately from Theorem 2 and Lemma 3.

With the result of Theorem 4, it is time to give the OPRemez algorithm for the design of linear phase FIR filters as follow.

The OPRemez algorithm for the design of linear phase FIR filters

(a) Specify the desired magnitude response, the weight function, the order of the filter and a dense grid of frequency.

(b) Take an initial set of $(r+1)$ extremal frequencies uniformly distributed in the passbands and stopbands of the filter;

(c) Calculate the coefficients $\alpha_0,\alpha_1,\cdots,\alpha_{r-1}$ and the error δ compacted in θ with the orthogonalized projection algorithm;

(d) Determine the new set of extremal frequencies of the weighted approximation error $E(\omega)$;

(e) If the new set of extremal frequencies is not equal to the initial set, then take the new set as the initial set and return to (c); Else, terminate.

OPRemez is a Remez exchange based algorithm. Theoretically it has the same convergence properties as the standard Remez exchange algorithm.

5. Design Examples

This section will give two design examples. The computation is done on a Pentium 166 computer and the design algorithm is implemented by the Object-Oriented Programming language Microsoft C++.

Example 1 To show the effectiveness of the OP-Remez algorithm, this example designs a high order low-pass filter with pass-band $[0, 0.19\pi]$ and stop-band $[0.21\pi, \pi]$. The weight function is 1 both in the pass-band and in the stop-band. The order of the filter is taken to be 351. Fig. 1 shows the weighted error curve with maximum error 6.92×10^{-4}. The design takes 9 exchanges and the design time is 59 seconds.

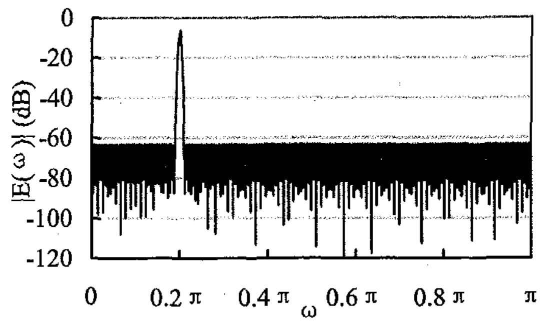

Fig. 1 The weighted error of the filter in example 1

Example 2 This example considers the design of a band-pass filter with one pass-band $[0.32\pi, 0.44\pi]$, and two stop-bands $[0, 0.28\pi]$ and $[0.48\pi, \pi]$. The weight function is 1.0 for all frequencies in the pass-band and stop-bands. Different filter orders are tested for this design. Table 2 depicts some properties (Ex. stands for exchanges and Sec. Stands for seconds) of the design for different filter orders.

Table 2 Design properties of example 2 for different orders

filter	OPRemez		Standard Remez		Both
order	Ex.	Sec.	Ex.	Sec.	δ
133	8	3	8	3	0.00355056
137	8	4	8	3	0.00293102
141	8	5	21	8	0.00240070
143	17	9	41	18	0.00199443
147	16	9	69	28	0.00198332
151	10	6	49	21	0.00194307
153	9	5	9	4	0.00181749

As a comparison, design results of the standard Remez exchange algorithm are also given in Table 2. It can be seen that the OPRemez algorithm takes no more exchanges for all designs, and takes much fewer exchanges and consumes much less time than the standard Remez exchange algorithm for some designs.

6. Conclusions and Remarks

This paper proposes an OPRemez algorithm for the minimax design of FIR filters. The OPRemez algorithm is a Remez exchange based algorithm in which the coefficients of the filter are evaluated with an orthogonalized projection algorithm. The estimated coefficients are proved to converge to the true values for a given set of extremal frequencies. Design examples show that the OPRemez algorithm can be used to design filters with very high order and that the algorithm obtains the same maximum error but takes fewer exchanges than the standard Remez exchange algorithm.

In OPRemez algorithm, the coefficients of the filter are obtained directly by solving linear equations for a given set of extremal frequencies. This has been applied recently by the author to the design of FIR filters with equation constraints in frequency domain.

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