# Robust Model Predictive Control for a Class of Uncertain Nonlinear Systems: An LMI Approach

Valiollah Ghaffari, Ali Akbar Safavi, Vahid Naghavi

*Abstract*— **This paper addresses a robust predictive control of additive discrete time uncertain nonlinear systems. The controller design is characterized as an optimization problem of the "worst-case" objective function over an infinite moving horizon. A sufficient state feedback synthesis condition is provided in the form of a linear matrix inequality (LMI) optimization and will be solved online at each time step. A simulation example is exploited to illustrate the applicability of the proposed approach.**

**Key words: LMI, Model predictive control, Uncertain nonlinear systems**

## I. INTRODUCTION

**M**PC techniques have been widely accepted by industry and academia. On other hand, because of existence of uncertainty in parameter or structure of processes, MPC strategy may fail. Thus robustness and performance of closed loop system may not satisfy.

Some algorithms with polytopic description have been proposed to solve state feedback robust MPC technique for stable linear systems with model uncertainty [1], norm bounded uncertain systems with input constraints [2] and output feedback robust MPC, such as constrained linear systems [3], bounded state disturbance and measurement noise [4], and additive but bounded state and output disturbances [5]. Constraints on the control effort (input) can be handled by adding another LMI to the LMI sets.

In [6] the concept of asymptotically stable invariant ellipsoid and linear matrix inequalities is used to developed an efficient on-line formulation of robust constrained MPC algorithm. In [7] disturbance model is included in controllers design to enhance the robustness of MPC to achieve offset-free control

Robust model predictive control has been used for constrained linear systems with bounded disturbances [8], linear continuous uncertain systems [9], linear continuous uncertain systems with state delay and control constraints [10]. For discrete-time uncertain state delayed systems a robust memory state feedback model predictive control is developed in [11].

Some well-known application of robust MPC are applied to CSTR problems [12-13], integrating systems at the presence of model uncertainty [14], and process with time-delay uncertainty like temperature control of a typical air-handling unit [15].

V. Ghaffari, A. A. Safavi, V. Naghavi are with the School of Electrical and Computer Engineering, Shiraz University, Iran (corresponding author ghaffari1363ir@ yahoo.com)

In these problems a polytopic structure is firstly developed to describe the uncertainty model. Then the controller design is characterized as the problem of minimizing an upper bound on the 'worst-case' infinite horizon objective function subject to constraints on the control input and plant output. Based on the proposed description, a linear matrix inequality (LMI) based MPC algorithm is employed and modified to design a robust controller for such a constraint process. The robust stability of the closed-loop Systems is guaranteed. In order to solve feasibility problem and assure system performance, some LMI conditions are proposed for the monotonical cost by using a new parameter dependent terminal weighting matrix [16].

They assume the system is linear. In this paper we extend these results to additive discrete time uncertain nonlinear systems. The controller design is characterized as an optimization problem of the "worst-case" objective function over infinite moving horizon. A sufficient state feedback synthesis condition is provided in the form of linear matrix inequality (LMI) optimization and will be solved online at each time step. A simulation example is exploited to illustrate the applicability of the proposed approach.

The rest of this article is organized as follows. In Section 2, some mathematical preliminaries are described. Section 3 presents an MPC control law for additive uncertain nonlinear systems. In order to demonstrate the validity of the approach a numerical example is presented in Section 4. Section 5 provides the concluding remarks.

## II. MATHEMATICAL PRELIMINARY

This section introduces 3 useful lemmas which are used in next section.

Lemma 1: Schur complement lemma: for any 3 matrix functions $Q(x)$, $S(x)$ and $R(x)$, the following inequalities are equivalent:

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0 \overset{if \quad and \quad only \quad if}{\Longleftrightarrow}$$

$$\begin{cases} R(x) > 0 \quad Q(x) - S(x)R^{-1}(x)S^T(x) > 0 \qquad (1) \\ \qquad\qquad OR \\ Q(x) > 0 \quad R(x) - S^T(x)Q(x)^{-1}(x)S(x) > 0 \end{cases}$$

Lemma 2: Barbalt's lemma

If $x(t) \in L_2$, $x(t) \in L_\infty$ and $\dot{x}(t) \in L_\infty$ then conclude that $\lim_{t \to \infty} x(t) = 0$.

Lemma 3: The following conditions are equivalent [17]:
   a. There exists a symmetric matrix P >0 such that

$$A^T PA - P < 0 \qquad (2)$$

b. There exist a symmetric matrix P and a matrix G such that

$$\begin{bmatrix} P & A^T G \\ G^T A & G + G^T - P \end{bmatrix} > 0 \qquad (3)$$

## III. MPC CONTROL LAW FOR ADDITIVE UNCERTAIN SYSTEM

Consider following discrete time dynamical system:

$$x(k+1) = Ax(k) + Bu(k) + f(x(k)) \qquad (4)$$

Where $A$ and $B$ are constant system matrixes and $f(x(k))$ is an unknown but Lipschitz bounded nonlinear term:

$$\|f(x(k))\| \le L\|x(k)\| \qquad f(0) = 0 \qquad (5)$$

$L$ is a positive real constant known number. A quadratic cost function may be given as:

$$J(k) = \sum_{i=0}^{\infty} x^T(k+i\,|\,k)Qx(k+i\,|\,k)$$
$$+ \sum_{i=0}^{\infty} u^T(k+i\,|\,k)Ru(k+i\,|\,k) \qquad (6)$$

Where $Q_{n\times n}$ and $R_{m\times m}$ are two positive definite states and control weights respectively. Now, consider a quadratic function like:

$$V(i,k) = x^T(k+i\,|\,k)Px(k+i\,|\,k)$$
$$P = P^T > 0 \qquad (7)$$

The objective is to obtain control sequence $u(k\,|\,k), u(k+1\,|\,k), ..., u(k+m\,|\,k)$ to minimize $J(k)$

$$\underset{u(k+i|k)\ \ i=0,1,...,m}{Min} J(k) \qquad (8)$$

Assume that the pair $(A, B)$ is stablizable by state feedback control law (i.e. There exists a matrix $F$ such that $A + BF$ is a stable matrix), then control effort value at time k and step i can be found as:

$$u(k+i\,|\,k) = F(k)x(k+i\,|\,k) \qquad (9)$$

Assume that $\Delta V(i,k) = V(i+1,k) - V(i,k)$ is less than a sum of two negative quadratic terms [18]:

$$\Delta V(i,k) < -x^T(k+i\,|\,k)Qx(k+i\,|\,k) + $$
$$-u^T(k+i\,|\,k)Ru(k+i\,|\,k) \qquad (10)$$

We use this inequality later on to obtain an LMI form. If $J(k)$ is a finite value, from Barbalat lemma, we know that $V(\infty, k) = 0$ then $\lim_{i\to\infty} x(i,k) = 0$. summing up $\Delta V(i,k)$ from zero to infinity, (i. e. $\sum_{i=0}^{\infty} \Delta V(i,k)$) an upper bound for cost function is computed as follows:

$$J(k) < V(0,k) = x^T(k\,|\,k)Px(k\,|\,k) < \gamma \qquad (11)$$

Defining $M = \gamma P^{-1}$ and using Schur complement lemma, the equivalent matrix form is found:

$$\begin{bmatrix} 1 & x^T(k\,|\,k) \\ x(k\,|\,k) & M \end{bmatrix} > 0 \qquad (12)$$

From this inequality it is seen why $\Delta V(i,k)$ must be less than sum of the two negative quadratic terms. We can rewrite $\Delta V(i,k)$ as:

$$\Delta V(i,k) < -\|x(k+i\,|\,k)\|_Q - \|u(k+i\,|\,k)\|_R < 0 \qquad (13)$$

Therefore

$$0 \le V(i+1,k) < V(i,k) \qquad (14)$$

Then Barbalat lemma leads us to $\lim_{i\to\infty} x(i,k) = 0$.

$\Delta V(i,k)$ can be written as:

$$\Delta V(i,k) = V(i+1,k) - V(i,k) =$$
$$[(A+BF(k))x(k+i\,|\,k) + f(x(k+i))]^T \times$$
$$P[(A+BF(k))x(k+i\,|\,k) + f(x(k+i))] +$$
$$-x^T(k+i\,|\,k)Px(k+i\,|\,k) +$$
$$< -x^T(k+i\,|\,k)Qx(k+i\,|\,k) +$$
$$-x^T(k+i\,|\,k)F(k)^T RF(k)x(k+i\,|\,k) \qquad (15)$$

We can rewrite (15) as:

$$\|(A+BF(k))x(k+i\,|\,k) + f(x(k+i))\|_P$$
$$-\|x(k+i\,|\,k)\|_P + \|x(k+i\,|\,k)\|_Q +$$
$$+\|F(k)x(k+i\,|\,k)\|_R < 0 \qquad (16)$$

Using lipschitz nonlinearity

$$\|f(x(k+i))\| \le L\|x(k+i)\| \qquad (17)$$

One may further simplify (16) to

$$(A+BF(k)+LI)^T P(A+BF(k)+LI) +$$
$$-P+Q+F(k)^T RF(k) < 0 \qquad (18)$$

Define $M = \gamma P^{-1}$ and $F(k) = YG^{-1}$ and using Schur complement lemma twice, we have (see [6], [18]):

$$\begin{bmatrix} G+G^T-M & * & * & * \\ AG+LG+BY & M & * & * \\ Q^{\frac{1}{2}}G & 0 & \gamma I & * \\ R^{\frac{1}{2}}Y & 0 & 0 & \gamma I \end{bmatrix} > 0 \qquad (19)$$

Where $FG = Y$ or $F = YG^{-1}$

The symbol * depicts a symmetric structure, and MPC problem is converted to optimization of an LMI problem. Equivalent LMI problem to above is:

$$\underset{Y,G,M}{Min} \ \gamma \qquad (20)$$

subject to

$$\begin{bmatrix} G+G^T-M & * & * & * \\ AG+LG+BY & M & * & * \\ Q^{\frac{1}{2}}G & 0 & \gamma I & * \\ R^{\frac{1}{2}}Y & 0 & 0 & \gamma I \end{bmatrix} > 0 \quad \text{and}$$

$$\begin{bmatrix} 1 & x^T(k\,|\,k) \\ x(k\,|\,k) & M \end{bmatrix} > 0$$

The gain of State feedback control law is $F(k)=YG^{-1}$. Since only first element of control sequence is applied to system, (i.e. $u(k\,|\,k)=F(k)x(k\,|\,k)$) then optimization problem is repeated.

Note that $x(k)=x(k\,|\,k)$ and $u(k\,|\,k)=u(k)$. In Figure 1 a schematic representation of yielded MPC is shown which act as a time varying state feedback control law (i.e. $u(k)=F(k)x(k)$) where $F(k)=YG^{-1}$ is computed at each sample time by an LMI optimization problem.
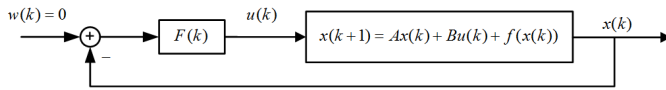


Fig 1: representation of MPC as a time varying state feedback

**Remark 1:** Time varying case
Consider time varying version of the mentioned system
$$x(k+1)=A(k)x(k)+B(k)u(k)+f(x(k)) \quad (21)$$
Where
$$[A(k)\,|\,B(k)]=\sum_{i=1}^{L}\lambda_i(k)[A_i\,|\,B_i]\ ,\ \sum_{i=1}^{L}\lambda_i(k)=1,\ \lambda_i(k)\geq 0$$

It can be easily seen that an LMI description is equivalent to:
$$\underset{Y,G,M}{Min}\quad \gamma$$
Subject to
$$\begin{bmatrix} G+G^T-M_j & * & * & * \\ A_jG+LG+B_jY & M_l & * & * \\ Q^{\frac{1}{2}}G & 0 & \gamma I & * \\ R^{\frac{1}{2}}Y & 0 & 0 & \gamma I \end{bmatrix} > 0 \quad (22)$$

$$\begin{bmatrix} 1 & x^T(k\,|\,k) \\ x(k\,|\,k) & M_j \end{bmatrix} > 0$$

$$\forall\quad j=1,2,...,L \quad \text{and} \quad \forall\quad l=1,2,...,L$$

**Remark 2**: Constrain on input and output [18].
a) One may bound on the Euclidean norm of the control signal using additional LMI
$$\begin{bmatrix} u_{max}^2 I & Y \\ Y^T & G+G^T-M_j \end{bmatrix} \geq 0 \quad (23)$$
$$\forall\quad j=1,2,...,L$$

b) One may bound on the Euclidean norm of the output signal using additional LMI
$$\begin{bmatrix} G+G^T-M_j & * \\ C(A_jG+B_jY) & y_{max}^2 I \end{bmatrix} \geq 0 \quad (24)$$
$$\forall\quad j=1,2,...,L$$

**Remark 3**: Tracking problem
Consider the system
$$x(k+1)=Ax(k)+Bu(k)+f(x(k))$$
$$y(k)=Cx(k) \quad (25)$$
To force the output $y(t)$ to track a reference signal $w(t)$, a cost function like
$$J(k)=\sum_{i=0}^{\infty}u^T(k+i\,|\,k)Ru(k+i\,|\,k)$$
$$+\sum_{i=0}^{\infty}\{(y(k+i\,|\,k)-w(k+i))^T Q\times$$
$$\times(y(k+i\,|\,k)-w(k+i))\} \quad (26)$$
must be minimized.

If this system is output feedback stablizable, then with the same procedure it can been shown that the resulted control law is $u(k+i\,|\,k)=F(k)(w(k+i)-y(k+i\,|\,k))$.

Note that $y(k)=y(k\,|\,k)$ and since only first computed control input is applied to plant (i.e. $u(k\,|\,k)=u(k)$), final control law is $u(k)=F(k)(w(k)-y(k))$ as illustrated in Figure 2. The result is a time varying static control law $F(k)=YG^{-1}$ so that $F(k)$ is computed at each sample time by an LMI optimization problem.
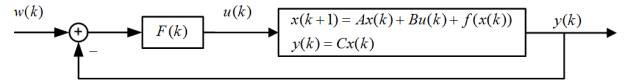


Fig 2: representation of MPC as a time varying output feedback

### IV. SIMULATION RESULT

To show the effectiveness of the proposed approach, we consider the discrete time dynamical system (4), where the linear terms are
$$A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.2 & 0.1 & 1 \end{bmatrix}, B=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{T=1 second}$$
and the nonlinear term is
$$f(x_k)=\begin{bmatrix} 0 & \dfrac{x_{1k}}{10+x_{3k}^2} & \dfrac{x_{2k}}{10+x_{3k}^2} \end{bmatrix}^T$$

When $L=0.1$, The Lipschitz nonlinearity is satisfied. This discrete time system starts from the initial condition $x_0=[-2\ -3\ 5]^T$. In cost function the weights are selected as $Q=I_{3\times3}, R=100$. The state response and control effort is seen in Figure 3 and 4 respectively.
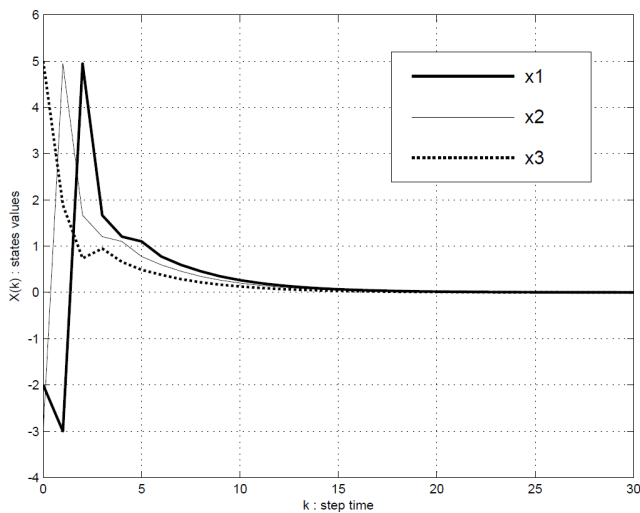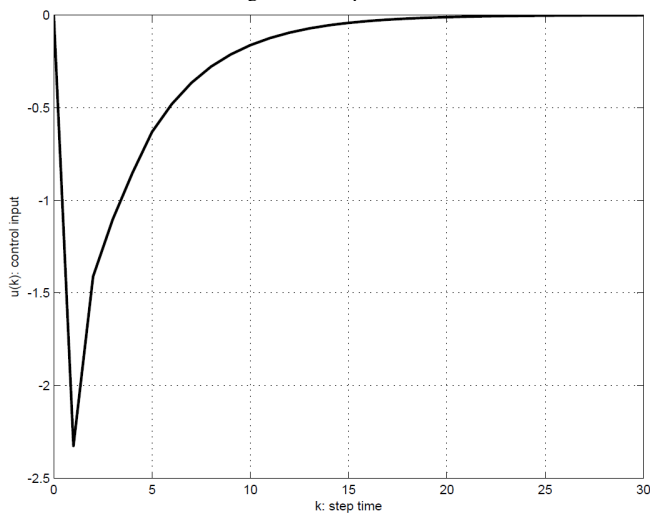
Fig 3: state response



Fig 4: applied control effort

In this example, similar to LQR problem, weights $R$, $Q$ are design parameter and can be selected such that the best response is obtained.

## V. CONCLUSION

In this paper based on LMI concepts, a time depended state feedback control law is proposed to control an additive discrete time uncertain systems. Each step by solving an LMI problem, a state feedback gain is computed. Then a suitable control effort is generated and applied to the dynamical system. Finally, the approach is applied to a nonlinear discrete time system which nonlinear term is norm bounded. Simulation results show the efficiency of the proposed approach.

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