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On Robust Adaptive Switched Control

Khalid El Rifai, Osamah El Rifai, and Kamal Youcef-Toumi
Massachusetts Institute of Technology
77 Massachusetts Ave. Room 3-348, Cambridge, MA 02139
elrifai@mit.edu, osamah@mit.edu, youcef@mit.edu

***Abstract*— In this paper, an algorithm for robust adaptive control design for a class of single-input-single-output (SISO) switched linear systems is developed. The control scheme guarantees robust exponential stability with respect to any bounded parametric uncertainty and bounded disturbance without requiring a priori knowledge of such bounds. Furthermore, switched system stability is preserved independent of switching speed for plant/controller parameter switching. The problem reduces to an analysis of an exponentially stable system driven by piecewise continuous inputs due to plant and controller parameter switching. This system is a modification of the closed loop error dynamics in standard adaptive control systems, through modifying the adaptation law. The results are illustrated for model reference adaptive control of a SISO linear plant.**

***Index Terms*— switched control, robust adaptive control, hybrid systems.**

I. Introduction

The control of systems characterized by hybrid, i.e., continuous and discrete, dynamics has been attracting many research efforts in recent years. This is motivated by the need to achieve reliably, repeatable, and safe control schemes to handel complex systems with switching dynamics of large, rapid, and sudden changes in model characteristics due to either natural (physical) changes or controlled (decision making based) changes. Such systems arise in many application such as robotics, communications, and chemical processes. The ability to achieve a priori stability and performance guarantees for switched and hybrid systems would not only extend the range of capabilities of control engineering practice but also allow for designing and operating systems at unprecedented levels of autonomy, versatility, and performance. There are two main issues with control of switched systems, which are stability and response of the switched system even when each subsystem is stable and known and the other is the robustness of stability with respect to uncertainty.

In terms of stability and response of switched systems, several results have been obtained in recent years, e.g. [5]. The most common approach to control of switched systems uses switching between linear-time-invariant (LTI) controllers. In this context, sufficient conditions for stability such as common Lyapunov functions and average dwell time [5] are the most commonly used tools. One class of results requires very sensitive adjustment to controller gains with each plant switch to guarantee stability for any switching speed. This is the case for common Lyapunov function based work, e.g., [8], [5], which requires switching control gains such that closed loop LTI system matrices are all stable and commute or are symmetric. Whereas another group of results shows stability if switching is slow on the average, e.g., [1], [5], which limits possible plant variations to be dealt with and requires gains to be adjusted to guarantee the stability of each frozen configuration and some level of knowledge of system parameters to compute the maximum admissible switching speed, which is the average dwell time.

The other problem of interest is that of dealing with uncertainty. The existing results are categorized based on the architecture being either that of a fixed robust control or standard adaptive control. In this regard, controller switching is used to deal with large uncertainties in a plant belonging to a known family of plants [1], [8], [6]. The first approach usually uses switching between LTI controllers, where stability is either based on a common Lyapunov function condition [8] or an average dwell time condition [1]. The weak robustness of LTI controllers with respect to parametric uncertainty causes the problem of unstable frozen plant/controller configuration [1]. On the other hand, methods based on adaptive control [6], [3] enjoy better stability guarantees in theory for similarly parameterized plants, if no disturbances or noise are present, analogous to standard adaptive control systems. Yet as standard adaptive control, these methods do not allow for characterizing the dynamic response neither for a frozen configuration nor for the overall dynamics. As suggested by the authors [6], [3], re-initializing the adaptation by switching between fixed estimates or resetting the adaptive estimate, is solely for improving transients, which is possible only if such fixed estimates are good. These results display similar pros and cons of the robust and adaptive methods analogous to those in non-switching control designs.

This paper takes a different approach to the control design of uncertain time varying switched systems. The approach used here attempts to combine the benefits of robust and adaptive control. This is achieved by using a standard adaptive control architecture, which is modified through the adaptation law to enforce exponential stability. Such a modification conflicts with adaptive control's philosophy since perfect tracking independent of parametric uncertainty, when no disturbances and time varying parameters are present, is lost. However, this theoretical result is given away for the sake of achieving pre-specified exponentially stable dynamic response and robustness with respect to any bounded time varying (including switching) disturbances and parametric uncertainties. In fact, this approach bypasses the switched system stability problem since switching in plant and controller parameters appear as step and impulse inputs to a continuous exponentially stable system. The results are demonstrated for model reference adaptive control (MRAC) of single-input-single-output (SISO) linear plants.

The remainder of the paper is organized as follows. Section II presents the stability of the modified adaptive control system. Section III presents the performance analysis of the switched control system. A simulation example is given in section IV based on MRAC for SISO linear-time-invariant (LTI) plants and conclusions are given in section V. The appendix provides a proof of the main result. In this paper, $\overline{\lambda}(.)$ and $\underline{\lambda}(.)$ denote the maximal and minimal eigenvalues of a matrix, $\|.\|$ the euclidian norm, and $diag(.,.,\dots)$ denotes a block diagonal matrix.

II. Stability of the Adaptive Controller

A. Standard Adaptive Control

Consider the adaptive control of *a minimum phase SISO LTI plant of known relative degree, upper bound on order, and sign of high frequency gain* [7], which can be either due to MRAC or a backstepping design. The corresponding closed loop dynamics commonly found in such problems is given by the form, e.g [7]:

$$\begin{aligned} \dot{e} &= A(t)e + B(t)Y(t)\tilde{a} + d(t) \\ \dot{\tilde{a}} &= -D(t)\Gamma E(t)Y(t)^T e \end{aligned} \quad (1)$$

where Γ is a constant symmetric positive definite gain matrix, $e = y - y_d$ is a tracking error vector and $\tilde{a} = \hat{a} - a$ is a parameter estimation error vector with constant parameters, $\dot{a} = 0$. Whereas $A(t)$, $B(t)$, $Y(t)$, $D(t)$, and $E(t)$ are known time varying matrices that depend on the control algorithm. Note that these matrices depend in general on the states of the system, which is suppressed here for notation simplicity. It is well known that such systems are Lyapunov stable systems with disturbance $d(t) = 0$. Using the quadratic Lyapunov function $V(e, \tilde{a})$ below:

$$V(e, \tilde{a}) = e^T P e + \tilde{a}^T \Gamma^{-1} \tilde{a} \quad (2)$$

shows that the system (1) is Lyapunov stable:

$$\dot{V}(e, \tilde{a}) = -2e^T C e \leq 0 \quad (3)$$

where C and P are appropriate constant symmetric positive definite matrices that depend on the control algorithm under consideration. Furthermore, the use of Barbalat's lemma would allow for concluding that $e \to 0$ as $t \to \infty$. Another approach to achieve robustness of stability and performance with respect to uncertainties is presented next, which is based on an adaptation law similar to leakage adaptive laws [2]. Note, however, that the approach used here does not rely, as in leakage and other robust adaptive methods, on adaptation law modification to bound perturbations (parameter time variation and disturbances) as internal terms so that the closed loop system remains Lyapunov stable.

B. The Modified Algorithm

Now consider the same system in Equation (1) with a modified adaptation law:

$$\begin{aligned} \dot{e} &= A(t)e + B(t)Y(t)\tilde{a} + d(t) \\ \dot{\tilde{a}} &= -D(t)\Gamma E(t)Y(t)^T e - L(\hat{a} - a^*) \end{aligned} \quad (4)$$

with L is a symmetric positive definite matrix and a^* is a chosen vector, which is an estimate of the plant parameter vector. The new system can be written as, for $\dot{a} \neq 0$:

$$\dot{x} = f(x,t) + \bar{v}(t)$$

where $x = [e, \tilde{a}]^T$ and $\bar{v} = [d, L(a^* - a) - \dot{a}]^T$. Then using the same Lyapunov function of Equation (2), we can see that the above system is composed of an unforced system $\dot{x} = f(x,t)$, which is of a globally exponentially stable fixed point $x = 0$. Furthermore, the forced system is BIBS stable following standard input-to-state stability arguments, see [4]. The following theorem gives a formal statement of the result, with the proof provided in the Appendix.

Theorem 1: Consider the system given by Equation (1) and the Lyapunov analysis of Equations (2-3) then the system given by Equation (4) is :
(i) Uniformly internally exponentially stable.
(ii) BIBS stable with
$\|e(t)\| \leq c_1 \|x(t_o)\| e^{-\alpha(t-t_o)} + c_2 \int_{t_o}^{t} e^{\alpha(\tau - t)} \|v(\tau)\| \, d\tau$.
where c_1, c_2 are constants, $\alpha = \bar{\lambda}(diag(P^{-1}C, L))$, and $v = [P^{1/2}d, \Gamma^{-1/2}(L(a^* - a) - \dot{a})]^T$.

C. Remarks

This section presents some remarks summarizing the implications of this result.

- The role of the extra term $-L(\hat{a}-a^*)$ in the adaptation law is to transform the Lyapunov stable homogenous adaptive control system into an exponentially stable system driven by an input $L(a^* - a)$.
- BIBS stability of the system guarantees boundedness of the states if $L(a^* - a)$, $d(t)$, and $\dot{a}$ are uniformly bounded (formally only need that $\int_{t_o}^{\infty} e^{\alpha(\tau - t)} \|v(\tau)\| \, d\tau < c$, where c is a constant).
- The system is therefore *robustly stable* with respect to any bounded magnitude parametric uncertainty and bounded disturbance *without requiring any a priori knowledge of such bounds*.
- Plant parameter switching no longer affects internal dynamics and stability but enters as a step change in input $L(a^* - a)$ and an impulse in input $\dot{a}$ at the switching instant.
- An allowed arbitrary time variation and switching in the parameter vector a suggests that such changes in the controlled plant parameters are for a plant with

the same assumed parameterized structure, i.e., same relative degree and order (or upper bound on order).

- Controller switching of a^* does not affect internal dynamics but enters as a step change in input $L(a^*-a)$.

III. Performance of the Control System

A. Dynamic Response

In this section, the performance of the system is evaluated. Since the effect of disturbances and parametric uncertainty and variations has been reduced to the effect of inputs onto a nominal homogenuous system, the response of the system is characterized by its response to inputs. In fact, due to exponential stability the response is *upper bounded by the response of an exponentially stable linear system* to these inputs. First consider the case with general time varying parameters and disturbances with $d(t), a(t), \dot{a}(t)$ being piecewise continuous uniformly bounded functions. This yields an upper bound on the input $v(t)$ given by $v_o = \sup_{t\geq t_o}\|v(t)\|$, which can be included in the expression in Theorem 1 (ii) to give:

$$\|e(t)\| \leq c_1\|x(t_o)\|e^{-\alpha(t-t_o)} + \frac{c_2}{\alpha}\, v_o(1-e^{-\alpha(t-t_o)})$$

with c_2/α being the attenuation factor of the input uncertainty $v(t)$ of size v_o. Whereas, one can see that convergence is made with settling time no slower than that dictated by the pre-specified decay rate α. Next, we consider the response to particular types of inputs.

1) Step and Impulse Inputs: First, considering impulsive changes in input $v(t)$ containing $d(t), a(t), \dot{a}(t)$, which more commonly arise due to switches in $a(t)$ yielding impulses in $\dot{a}(t)$. Due to exponential stability of this system, the response to impulses is very similar to that of linear systems. In fact, the impulses in input $v(t)$ with a change v_p (size of impulse) can be represented with dirac-delta function, i.e., $\|v(t)\| \leq v_p\,\delta(t-t_i)$ at impulse times t_i. This yields the following response to impulse inputs after evaluating the integral:

$$\|e(t)\| \quad \leq \quad \begin{cases} c_1\|x(t_o)\|e^{-\alpha(t-t_o)} + c_2\, v_p e^{-\alpha(t-t_i)} \\ c_1\|x(t_o)\|e^{-\alpha(t-t_o)} \qquad ,t<t_i \end{cases}$$

where the 2^{nd} term in the right hand side vanishes exponentially after approaching the upper bound $c_2 v_p$. Whereas for a step input in $v(t)$, which arises either in d or in switches in a or a^*, we denote the norm of $v(t)$ after a step by $v_s = \sup_{t\geq t_i}\|v(t)\|$, which is the size of the step change and thus:

$$\|e(t)\| \quad \leq \quad \begin{cases} c_1\|x(t_o)\|e^{-\alpha(t-t_o)} + \frac{c_2}{\alpha}\, v_s\,(1-e^{-\alpha(t-t_i)}) \\ c_1\|x(t_o)\|e^{-\alpha(t-t_o)} \qquad ,t<t_i \end{cases}$$

2) Frequency Response: The last type of input to be detailed here is sinusoidal inputs, which allows for capturing a frequency dependent response analogous to that in linear control but in a more primitive form. In this case we have $\|v\| \leq |A\, sin(\omega t)|$. Carrying out the integration which yields a messy algebraic expression for the time dependent response. In the spirit of linear analysis, we will focus on the steady state response for this input, which leads to the following upper bound:

$$\|e(t)\| \quad \leq \quad c_1\|x(t_o)\|e^{-\alpha(t-t_o)} + c_2\frac{A}{\alpha}\left(\frac{1+\bar{\omega}}{1+\bar{\omega}^2}\right)$$

An interesting observation is that the term due to the sine input on the right hand side, though a conservative upper bound, captures the essence of frequency response. This can be seen in the attenuation factor of the sine of amplitude A, which has been re-written in terms of $\bar{\omega} = \omega/\alpha$, the normalized frequency of the input. By noting that the decay rate α plays the role of system bandwidth (slowest eigenvalue) we can see that $\bar{\omega}$ is a measure of the frequency of the input relative to the system's bandwidth. One can see that the term $(1+\bar{\omega})/(1+\bar{\omega}^2) \approx 1$ for $0 < \bar{\omega} \leq 1$. This means that for inputs within system bandwidth the attenuation of such inputs scales with the system bandwidth α. Whereas for $\bar{\omega} > 1$ the attenuation factor drops towards zero as $\bar{\omega}$ is increased. This implies that out of bandwidth inputs are attenuated independent of the system's gain with this attenuation increasing with the frequency of the input. These two properties are the essence of low and high frequency disturbance attenuation commonly used in linear control. The results extend directly to combinations of sinusoidal inputs with different frequencies and phases by superposition of the integration.

B. Improving Tracking Error

Since stability and dynamic response of the system to different inputs and uncertainties have been established independent of uncertainty, we are now left with optimizing the control parameters and gains a^*, L, Γ, P, and C for minimal tracking error. An informal discussion is made here on the effect of the "size" of parameters, by which we mean the induced norm $\|(.)\|$ for a matrix. There are several means for improving tracking error for a system subject to large disturbances and uncertainties, which can be employed depending on the modeling and control trade-offs associated with each problem. These methods are:

1) *Increasing the system input-output gain* $\alpha = \bar{\lambda}(diag(P^{-1}C, L))$, which as discussed earlier, acts on the the overall input uncertainty v. This attenuation, however, increases the system bandwidth, which suggests its use primarily for low/high bandwidth disturbances along the line of frequency response analysis of last section.
2) *Increasing adaptation gain* Γ, which has the effect of attenuating parametric uncertainty independent of system bandwidth (Recall that α is independent of Γ from Theorem 1). This is the case since the size of the input v is reduced by reducing the component $\Gamma^{-1/2}(L(a^*-a)-\dot{a})$. Note that a very large Γ has the effect of amplifying measurement noise, which can be seen from the adaptation law.

3) *Using a small gain* $\Gamma^{-1/2}L$, which is an agreement with increasing adaptation gain matrix Γ mentioned above. However, this differs by the fact that this can be also achieved by simply reducing the size of L. Furthermore, using $\Gamma^{-1/2}L$ is effective mainly for constant parametric uncertainty since the input v contains $\Gamma^{-1/2}(L(a^* - a) - \dot{a})$, which suggests a small $\Gamma^{-1/2}L$ does not necessarily attenuate $\dot{a}$ unless $\Gamma^{-1/2}$ is also small. This is the case since this condition implies having *approximate integral action* in the adaptation law of Equation (4), i.e., approaching integral action in the standard gradient adaptation law.
4) *Adjusting and updating parameter estimate* a^* , which can be any piecewise continuous bounded function. This allows for reducing the effect of parametric uncertainty through reducing size of input $a^* - a$ independent of system bandwidth and control gains. In this regard, if the plant parameters a are known to belong to a set (or range) then we can employ switching and estimation ideas to find a better guess of a^* if the initial guess did not yield good performance. Furthermore, if the plant parameters change significantly with time, e.g. switch, then updating a^* by mode specific nominal estimate(s) of a can allow for quickly reducing the tracking error.

C. Remarks

- Exponential stability allows for shaping the transient response, e.g. settling time, and frequency response of the system to low/high frequency dynamics and inputs by adjusting the decay rate α, see Theorem 1. This is to be done independent of the parametric uncertainty $a^* - a$, which is contrasted to LTI feedback where closed loop poles change with parametric uncertainty.
- The attenuation of uncertainty by high input-output system gain in this scheme differs from robust control by the fact that BIBS stability, the pre-requisite to such attenuation, is never lost due to large parametric uncertainty $a^* - a$. This is the case since it no longer enters as a function of the plant's state but rather as an input $L(a^* - a)$.
- In switching between different a^* values many of the useful and interesting ideas to monitor, select, and switch between different candidate controllers (parallel estimation, localization, etc) such as those in [1], [6], [8] can be used with a_i^* values playing the role of the i^{th} candidate controller. The difference is that this is to be done without frozen-time instability or switched system instability concerns (verifying dwell time or common Lyapunov function conditions). Furthermore, since the dynamic response is guaranteed by design the pre-specified settling time can be utilized to quickly decide on the adequacy of implemented controllers.
- For control switches of a^*, the actual control signal is smooth, which is attractive for practical implementation. This is the case since a^* only enters in the adaptation law with $\dot{\hat{a}}$ piecewise continuous and thus $\hat{a}$ is smooth and the control signal is given by $u(t) = W(t)\hat{a}$ or equivalent.

IV. Example Simulation

In this section, case study simulations are shown to demonstrate the key results presented in this paper. Consider a MRAC, see [7], [4] for control design details, for the following unstable 2^{nd} order plant of relative degree 1:

$$\begin{aligned} \dot{x}_1 &= b_1x_1 + b_2x_2 + b_3u + b_3d \\ \dot{x}_2 &= x_1 \\ y &= c_1x_1 + c_2x_2 + n \end{aligned} \tag{5}$$

where $b_1 = 3$, $b_2 = -2$, and $c_1 = c_2 = b_3 = 1$ are the nominal simulation values for which we will denote the vector a as parameterized vector corresponding to these values. Whereas, u, d, and n are control signal, disturbance, and measurement noise respectively. The measurement noise used in the simulations is of SN $1 : 1000$ and a pair of unmodeled complex poles at 22 rad/sec and 0.07 damping ratio are also included. The reference model used is of the following transfer function:

$$W_m(s) = \frac{a_m}{s + a_m}$$

Let us choose the nominal $a_m = 1$ and $L = I$, where I is the identity matrix, then we have from Theorem 1 that the decay rate $\alpha = 1$ rad/sec, since $P^{-1}C = a_m$ in this case. This should yield a settling time of about 4 seconds for the closed loop system. Also the nominal value of the adaptation gain Γ will be denoted $\Gamma_o = 100I$.

Figure 1 shows the response of MRAC for the output of the plant tracking a sinusoidal reference of amplitude 2 and frequency 0.3 rad/sec. The standard MRAC, denoted standard, shows poor and unpredictable transients, which were also observed when noise and unmodeled dynamics were removed but with less oscillations. The response of the modified algorithm is shown for $a^* = 10a$, i.e. 900% parametric uncertainty, with the same adaptation gain $\Gamma_o = 100$, yielding predesigned settling, 4 seconds for $\alpha = 1$, and better transients. The steady state tracking error was clearly nonzero, unlike the standard controller, yet the error is practically zero with maximum error of about 2% of the size of the reference. The response of the modified algorithm for a much larger parametric uncertainty $a^* = 100a$ is shown for the nominal adaptation gain and for an increased adaptation gain $\Gamma = 10\Gamma_o$. For the larger uncertainty the error grows to about 15% of the reference yet an increase in Γ yields a tracking error almost identical to the lower uncertainty case with lower adaptation gain. Similar results were observed by reducing L, which are along the lines of the discussion in Section III.B.

Figure 2 shows the response of the control signal for the cases in Figure 1. The control signal for the standard adaptive controller is clearly very aggressive during the transients. Whereas, for the modified algorithm, the control signal for larger Γ values is essentially the same as with low Γ since the increase in Γ is balanced by a decrease in tracking error.

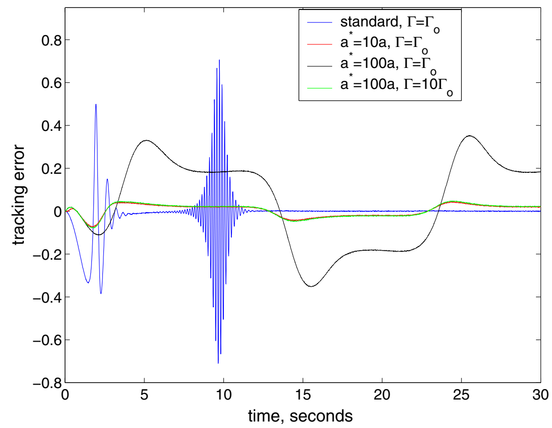

Fig. 1. Tracking error comparison of standard and modified MRAC for the nominal LTI plant.

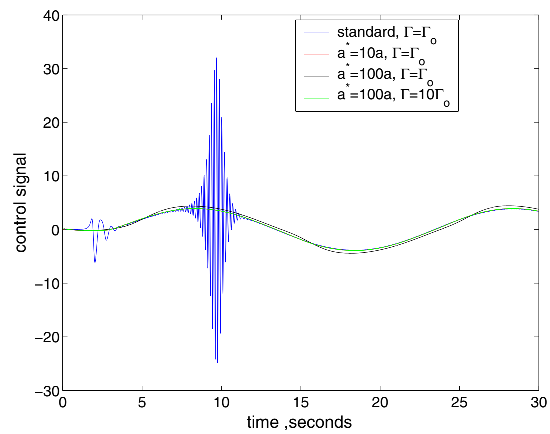

Fig. 2. Control signal for comparison of standard and modified MRAC for the nominal LTI plant.

The situation in Figure 1 is repeated but with time varying parameters in Figure 3. First, the fixed plant parameters b_1, b_2 in Equation (5) are replaced with time varying ones $(1 + 2\cos(5t))b_i$ where b_i are nominal constant values for $i = 1, 2$. The estimate $a^* = 10a$ is used with a denoting the nominal values of the LTI plant used in Figures 1 and 2. The response is shown, denoted $a(t)$ small, to yield small tracking error of 5% for the nominal control values. Then the time variation is increased by using $(1 + 4\cos(5t))b_i$ instead of b_i where b_i are nominal constant values for $i = 1, 2$. This corresponds to an increase in tracking error to about 15% of the reference. Yet a 10 times increase in Γ reduces the error to a much smaller value of about 2%, see '$a(t)$ big' plots in Figure 3.

In Figure 4, the response for the nominal control parameters and $a^* = 10a$ with a sinusoidal disturbance $d = 10\sin(2\pi t)$ is shown. The tracking error is clearly larger than that with no disturbance, Figure 1. This is the case since the attenuation factor due to the closed loop bandwidth, see Frequency response analysis in Section III.B, is not good enough. Yet an increase in bandwidth $\alpha = 10$, for $a_m = 10$, $L = 10I$, and $\Gamma = 10\Gamma_o$, yields better disturbance attenuation to an error within 2% of the reference. Whereas, using the original low system bandwidth and increasing the frequency of the disturbance yields disturbance attenuation due to system roll-off. In fact, the response for the case where $\alpha = 1$ and $d = 10\sin(2\pi 1000t)$ is almost indistinguishable from that without the disturbance in Figure 1 for the same $a^* = 10a$.

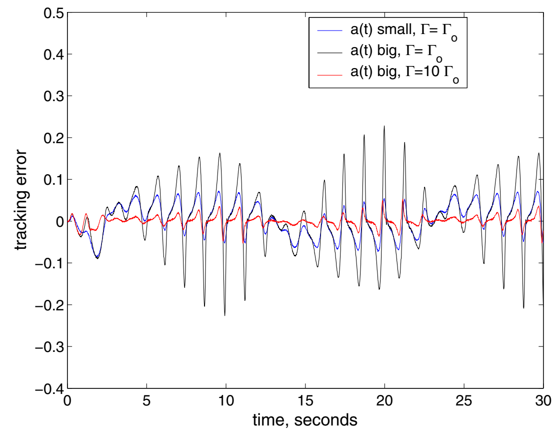

Fig. 3. Tracking error for modified MRAC for an LTV plant with $a^* = 10a$.

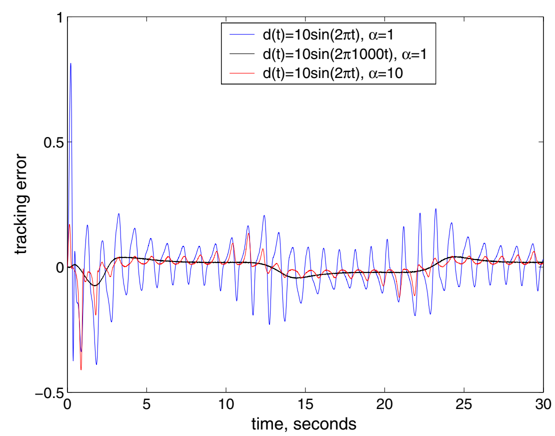

Fig. 4. Tracking error for modified MRAC for the nominal LTI plant for $a^* = 10a$ with disturbances.

Next, we consider some plant and controller switching scenarios. Figure 5 shows 3 plant switches from the nominal value of b_1 to $2b_1$, $-2b_1$, and $2b_1$ at times $t = 10, 14, 21$ seconds and similarly for b_2. The system responds to the corresponding impulse change in $\dot{a}$ and step change in $a^* - a$ with the error remaining small. However, a 4^{th} plant switch to $12b_1$ and $12b_2$ at $t = 25$ seconds yields a much bigger jump in error due to the impulse in $\dot{a}$ being large, yet the error quickly settles to a small value since the actual $a^* - a$ after the switch is attenuated. Whereas, a fifth switch to the plant at $t = 35$ seconds yields a large jump in error, about 10%, followed by settling to relatively large tracking error. This is followed by two controller switches of a^* values at $t = 40, 50$ to improve tracking error. Following the first controller switch, the tracking error is made smaller. Whereas, controller switching stops after the second switch yielding no further improvement in tracking error, which settles to within 2% of the reference. The parameter estimates are also shown in 5 (b), which displays the expected response to step and impulse inputs corresponding to switches in a and a^*.

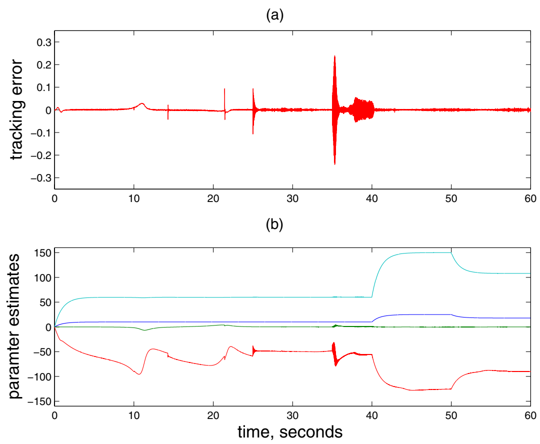

Fig. 5. (a) Tracking error and (b) parameter estimates, for modified MRAC for switching linear plant.

V. Conclusions

An algorithm for stable robust adaptive control for SISO switched linear systems has been presented. The control scheme guarantees robust exponential stability with respect to any bounded parametric uncertainty and bounded disturbance without requiring a priori knowledge of such bounds. Furthermore, switched system stability is preserved independent of switching speed for all considered plant/controller switching scenarios. The problem has been reduced to an analysis of a continuous system with switching inputs. This system is a modification of the closed loop error dynamics in continuous adaptive control, through modifying the adaptation law. The results are demonstrated for MRAC of SISO linear plants. Future work will detail the application of the idea to further classes of systems and investigate further switching control capabilities.

Appendix

A. Proof of Theorem 1

The proof follows standard exponential stability and BIBS stability arguments [4], but is slightly modified to yield less conservative bounds by taking advantage of the quadratic nature of the Lyapunov function.

Proof:
Let $x = [e, \tilde{a}]^T$ and $z = Sx$, where $S = diag(P^{1/2}, \Gamma^{-1/2})$ a symmetric positive definite matrix. Using the Lyapunov function $V(e, \tilde{a}) = e^T P e + \tilde{a}^T \Gamma^{-1} \tilde{a}$ and the result from Equation (3), we have for the system given by Equation (4):

$$\begin{aligned} \dot{V}(x) &= -2x^T SMSx + 2x^T S^2 \bar{v} \\ &= -2z^T Mz + 2z^T v \end{aligned}$$

where $M = diag(P^{-1/2}CP^{-1/2}, \Gamma^{-1/2}L\Gamma^{1/2})$, $\bar{v} = [d, L(a^* - a) - \dot{a}]^T$ and $v = S\bar{v}$. But we have $V = \|z\|^2$, which means:

$$\frac{1}{2}\frac{d}{d\tau}\|z\|^2 = -z^T Mz + z^T v \le -\alpha\|z\|^2 + \|z\|\|v\|$$

where $\alpha = \underline{\lambda}(M) = \underline{\lambda}(diag(P^{-1}C, L))$ by similarity. Hence

$$\|z\|\left(\frac{d}{d\tau}\|z\| + \alpha\|z\| - \|v\|\right) \le 0$$

Therefore either $\|z\| = 0 \; \forall \tau$, which is clearly not the case, or we have:

$$\frac{d}{d\tau}\|z\| \le -\alpha\|z\| + \|v\|$$

Integrating and re-arranging we then have:

$$\|z(t)\| \le \|z(t_o)\|e^{-\alpha(t-t_o)} + \int_{t_o}^{t} e^{\alpha(\tau-t)}\|v(\tau)\|\,d\tau$$

By definition of $\|z\| = \|Sx\|$ we can get that:

$$\|x(t)\| \le c_1\|x(t_o)\|e^{-\alpha(t-t_o)} + c_2\int_{t_o}^{t} e^{\alpha(\tau-t)}\|v(\tau)\|\,d\tau$$

with $c_1 = \|S\|\|S^{-1}\|$ and $c_2 = \|S^{-1}\|$. Internal exponential stability is shown by letting $v = 0$ above, which proves part (i). BIBS stability is shown by denoting $v_o = \sup_{t \ge t_o}\|v(t)\| < \infty$, then from the last expression we have:

$$\|x(t)\| \le c_1\|x(t_o)\|e^{-\alpha(t-t_o)} + \frac{c_2}{\alpha}v_o$$

The required expression in part (ii) follows directly from the one above with $\|e\| \le \|x\|$. ■

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