International Conference on Control, Automation and Systems 2010
Oct. 27-30, 2010 in KINTEX, Gyeonggi-do, Korea

Sliding mode control design under partial state feedback for ball and beam system

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Abstract: In this paper, we develop a state observer in order to estimate the velocity of a ball and angular velocity of a motor. With the observer, we propose a sliding mode controller where sliding surface consists of some measured and estimated states. Then, a new rigorous analysis will be done to show the behavior of the system on the sliding surface. We show the stability analysis and convergence analysis of the partial state observer. In addition, the experimental results show the validity of the proposed controller.

Keywords: Partial state feedback, Sliding mode control, Ball and beam system

1. INTRODUCTION

In recent years, many contributions have been presented in control literature that solve the control design problem for classes of nonlinear systems [8]. The ball and beam system is one of the most popular models for studying control systems because of its nonlinearity and several control techniques have been proposed [1], [3] - [4]. In many of results, all state variables of system are required to be available. However, some state variables may not be measured physically or some sensors may be too expensive to use. In order to overcome this actual control problem, some state observers are often utilized. Jo and Seo proposed a global nonlinear observer that guarantees the estimation error converge to zero asymptotically [5]. Also, Krener and Isidori proposed the Lie-algebraic conditions under which nonlinear observers with linearizable error dynamics can designed [6]-[7].

In this paper, we propose a sliding mode control under partial state feedback for ball and beam system. We use the Quanser's ball and beam system whose sensors only can measure the position of ball and the angle of motor [9]. Thus, we need to develop an observer to estimate ball and angular velocity. In our controller, the sliding surface consist of both of measured and estimated state. We verify the validity of our proposed method through experimental results.

2. MODELING

The equation of motion describing the ball and beam system can be written as [9].

$$\begin{aligned}\ddot{x} &= K_{bb}\sin\theta \\ \ddot{\theta} &= -\frac{1}{\tau}\dot{\theta} + \frac{K}{\tau}V_m\end{aligned} \qquad (1)$$

where the gain K_{bb} of the system is given by

$$K_{bb} = \frac{m_b g r_{arm} r_b^2}{L_{beam}(m_b r_b^2 + J_b)}$$

where x position of ball, θ angle of motor, K steady-state gain, τ time constant, m_b mass of ball, g gravitational constant, r_{arm} distance between SRV02 output gear shaft, r_b radius of ball L_{beam} beam length, J_b moment of inertia of ball.

We let the states of the system be as follows

$$x = x_1, \quad \dot{x}_1 = x_2, \quad \theta = x_3, \quad \dot{\theta} = x_4 \qquad (2)$$

Then, the state equation is given by (3)

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= K_{bb}\sin x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -\frac{1}{\tau}x_4 + \frac{K}{\tau}V_m\end{aligned} \qquad (3)$$

Here, x_1 (position of ball) and x_3 (angle of motor) are measured by using the BB01 potentiometer sensor and tachometer. However, x_2 (ball velocity) and x_4 (angular velocity) are unmeasured. So, we need to estimate these variables x_2, x_4. Therefore, we propose an observer for unmeasured variables.

3. STATE OBSERVER DESIGN

In this section, we propose a nonlinear observer for the ball and beam system. We reconfigure the system (3) as follows

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= K_{bb}x_3 + \underbrace{K_{bb}\sin x_3 - K_{bb}x_3}_{\delta_2(t,x,u)} \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \underbrace{-\frac{1}{\tau}x_4}_{\delta_4(t,x,u)} + \frac{K}{\tau}V_m\end{aligned} \qquad (4)$$

Hence, the state space equation is

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & K_{bb} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K}{\tau} \end{bmatrix}}_{B} V_m$$

$$+\underbrace{\begin{bmatrix} 0 \\ K_{bb}\sin x_3 - K_{bb}x_3 \\ 0 \\ -\frac{1}{\tau}x_4 \end{bmatrix}}_{\delta(t,x,u)}$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}}_{C} x \quad (5)$$

The nonlinear observer is proposed as [2],

$$\dot{z} = Az + Bu - L(\epsilon_L)(y - Cz) + \delta(t,z,u) \quad (6)$$

where $L(\epsilon_L) = [\frac{l_1}{\epsilon_L}, \cdots, \frac{l_n}{\epsilon_L^n}]^T$ with $\epsilon_L > 0$ and $\delta(t,z,u) = [0, \delta_2(t,z,u), 0, \delta_4(t,z,u)]$. Therefore,

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & K_{bb} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K}{\tau} \end{bmatrix} V_m - \begin{bmatrix} \frac{l_1}{\epsilon_L} \\ \frac{l_2}{\epsilon_L^2} \\ \frac{l_3}{\epsilon_L^3} \\ \frac{l_4}{\epsilon_L^4} \end{bmatrix} (y - Cz) + \underbrace{\begin{bmatrix} 0 \\ K_{bb}\sin z_3 - K_{bb}z_3 \\ 0 \\ -\frac{1}{\tau}z_4 \end{bmatrix}}_{\delta(t,z,u)} \quad (7)$$

3.1 Analysis of observer stability

Defining the error $e = x - z$, we can write the error dynamics as

$$\dot{e} = A_L(\epsilon_L)e + \delta(t,x,u) - \delta(t,z,u) \quad (8)$$

where $A_L(\epsilon) = A + L(\epsilon_L)C$, we denote $E(\epsilon_L)$ =diag$[1, \epsilon_L, \cdots, \epsilon_L^{n-1}]^T$, $A_L = A + LC$ ($L = [l_1, \cdots, l_n]$). Then, we have the following equation

$$\epsilon_L A_L(\epsilon_L) = E(\epsilon_L)^{-1} A_L E(\epsilon_L) \quad (9)$$

where $A_L(\epsilon) = A + L(\epsilon_L)C$ and $A_L = A + LC$ are Hwritz. By Lyapunov equation $A_L^T P_L + P_L A_L = -I$, we obtain

$$A_L^T(\epsilon_L)P_L(\epsilon_L) + P_L(\epsilon_L)A_L(\epsilon_L) = -\epsilon_L^{-1}E(\epsilon_L)^2$$

$$P_L(\epsilon_L) = E(\epsilon_L)P_L E(\epsilon_L) \quad (10)$$

We set $V_o(e) = e^T P_L(\epsilon_L)e$ for (8). Then, along the trajectory of (8),

$$\begin{aligned} \dot{V}_o(e) &= \dot{e}^T P_L(\epsilon_L)e + e^T P_L(\epsilon_L)\dot{e} \\ &= -\epsilon_L^{-1}||E(\epsilon_L)e||^2 \\ &\quad +2e^T E(\epsilon_L)P_L E(\epsilon_L)[\delta(t,x,u) - \delta(t,z,u)] \\ &\leq -\epsilon_L^{-1}||E(\epsilon_L)e||^2 + 2||P_L||||E(\epsilon_L)e|| \\ &\quad \times||E(\epsilon_L)[\delta(t,x,u) - \delta(t,z,u)]|| \end{aligned} \quad (11)$$

We consider $K_{bb}\sin z_3 - K_{bb}z_3 = K_{bb}\sin x_3 - K_{bb}x_3$ from the equation (5) and (7) because x_3 angle of motor can be measured using the sensor. Thus, we obtain $||E(\epsilon_L)\{\delta(t,x,u) - \delta(t,z,u)\}|| \leq \frac{1}{\tau}||E(\epsilon_L)e||$. Finally, we have

$$\dot{V}_o(e) \leq -N_o||E(\epsilon_L)e||^2, \; N_o = \epsilon_L^{-1} - \frac{1}{\tau}2||P_L|| \quad (12)$$

$\dot{V}_o(e)$ is negative definite when $N_o = \epsilon_L^{-1} - \frac{1}{\tau}2||P_L|| > 0$. The global exponentially stability easily follows from the equation (12).

3.2 Analysis of observer convergence

We reconsider

$$V_o(e) = e^T P(\epsilon_L)e = e^T E(\epsilon_L)P_L E(\epsilon_L)e \quad (13)$$

We obtain following the equation (14) from the equation (13).

$$\lambda_{\min}(P_L)||E(\epsilon_L)e||^2 \leq V_o(e) \leq \lambda_{\max}(P_L)||E(\epsilon_L)e||^2 \quad (14)$$

By the equation (12) and (14)

$$V_o(e) \leq V_o(0)e^{-\frac{N_o}{\lambda_{\max}(P_L)}t} \quad (15)$$

Thus, by the equation (14) and (15)

$$\lambda_{\min}(P_L)||E(\epsilon_L)e||^2 \leq V_o(e) \leq V_o(0)e^{-\frac{N_o}{\lambda_{\max}(P_L)}t} \quad (16)$$

Equation (16) can be rewritten as (17)

$$||E(\epsilon_L)e||^2 \leq \frac{\lambda_{\max}(P_L)}{\lambda_{\min}(P_L)}||E(\epsilon_L)e(0)||^2 e^{-\frac{N_o}{\lambda_{\max}(P_L)}t} \quad (17)$$

We finally obtain

$$||E(\epsilon_L)e|| \leq \sqrt{\frac{\lambda_{\max}(P_L)}{\lambda_{\min}(P_L)}}||E(\epsilon_L)e(0)||e^{-\frac{N_o}{2\lambda_{\max}(P_L)}t} \quad (18)$$

We find that the observer error can rapidly converge to zero by scaling factor ϵ_L under the condition of $N_o = \epsilon_L^{-1} - \frac{1}{\tau}2||P_L|| > 0$.

4. SLIDING MODE CONTROL DESIGN

Define the regulation errors

$$\begin{aligned} e_\theta &= \theta - \theta_e = \theta \\ e_x &= x - x_d \end{aligned} \quad (19)$$

where $\theta_e = 0$ and x_d is a constant desired value. Also, define the sliding surface

$$\begin{aligned} s &= \dot{e}_\theta + \frac{\beta_3}{\epsilon_K}e_\theta + \frac{\beta_2}{\epsilon_K^2}\dot{e}_x + \frac{\beta_1}{\epsilon_K^3}e_x \\ &= \dot{\theta} + \frac{\beta_3}{\epsilon_K}\theta + \frac{\beta_2}{\epsilon_K^2}\dot{x} + \frac{\beta_1}{\epsilon_K^3}(x - x_d) \\ &= x_4 + \frac{\beta_3}{\epsilon_K}x_3 + \frac{\beta_2}{\epsilon_K^2}x_2 + \frac{\beta_1}{\epsilon_K^3}(x_1 - x_d) \end{aligned} \quad (20)$$

Differentiating s in (20) with respect to time, we obtain

$$\begin{aligned} \dot{s} &= \dot{x}_4 + \frac{\beta_3}{\epsilon_K}\dot{x}_3 + \frac{\beta_2}{\epsilon_K^2}\dot{x}_2 + \frac{\beta_1}{\epsilon_K^3}\dot{x}_1 \\ &= -\frac{1}{\tau}x_4 + \frac{K}{\tau}V_m + \frac{\beta_3}{\epsilon_K}x_4 \\ &\quad + \frac{\beta_2}{\epsilon_K^2}K_{bb}\sin x_3 + \frac{\beta_1}{\epsilon_K^3}x_2 \end{aligned} \quad (21)$$

where V_m is a control input to the ball and beam system. Hence,

$$\begin{aligned} V_m &= \frac{\tau}{K}\left(\frac{1}{\tau}x_4 - \frac{\beta_3}{\epsilon_K}x_4 - \frac{\beta_2}{\epsilon_K^2}K_{bb}\sin x_3 \right. \\ &\quad \left. - \frac{\beta_1}{\epsilon_K^3}x_2 - \Gamma\text{sgn}(s)\right) \end{aligned} \quad (22)$$

In this time, we can use z_2 and z_4 instead of x_2 and x_4 because we know that observer error can rapidly converge to zero by the equation (18). Thus we reconfigure the system (22) as follows

$$V_m \approx \frac{\tau}{K}\left(\frac{1}{\tau}z_4 - \frac{\beta_3}{\epsilon_K}z_4 - \frac{\beta_2}{\epsilon_K^2}K_{bb}\sin x_3 - \frac{\beta_1}{\epsilon_K^3}z_2 - \Gamma\text{sgn}(s)\right) \tag{23}$$

Substituting the controller given by (22) into the above equation, it follows that

$$\dot{s} = -\Gamma\text{sgn}(s) \tag{24}$$

The equation in (24) guarantees that $s\dot{s} < 0$
The system on the sliding surface $s = 0$ is given by

$$z_4 = -\frac{\beta_3}{\epsilon_K}x_3 - \frac{\beta_2}{\epsilon_K^2}z_2 - \frac{\beta_1}{\epsilon_K^3}(x_1 - x_d) \tag{25}$$

Thus, the reduced order system is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= K_{bb}\sin x_3 \\ \dot{x}_3 &= -\frac{\beta_3}{\epsilon_K}x_3 - \frac{\beta_2}{\epsilon_K^2}z_2 - \frac{\beta_1}{\epsilon_K^3}(x_1 - x_d) \end{aligned} \tag{26}$$

Since $z_2 = x_2 - e_2$, we can write

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= K_{bb}\sin x_3 \\ \dot{x}_3 &= -\frac{\beta_3}{\epsilon_K}x_3 - \frac{\beta_2}{\epsilon_K^2}x_2 - \frac{\beta_1}{\epsilon_K^3}(x_1 - x_d) + \frac{\beta_2}{\epsilon_K^2}e_2 \end{aligned} \tag{27}$$

Here, e_2 is vanishing term because $e \to 0$. The system in (27) can be written in compact form as

$$\dot{x} = f_1(x) \tag{28}$$

where

$$f_1 = \begin{bmatrix} x_2 \\ K_{bb}\sin x_3 \\ -\frac{\beta_3}{\epsilon_K}x_3 - \frac{\beta_2}{\epsilon_K^2}x_2 - \frac{\beta_1}{\epsilon_K^3}(x_1 - x_d) \end{bmatrix} \tag{29}$$

We will linearize the system in (28) around the origin $x = 0$. Therefore, we obtain the following linearized system around the origin,

$$\dot{x} = A_1(\epsilon_K)x \tag{30}$$

where

$$\begin{aligned} A_1(\epsilon_K) &= \left.\frac{\partial f_1(x)}{\partial x}\right|_{x=0} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & K_{bb} \\ -\frac{\beta_1}{\epsilon_K^3} & -\frac{\beta_2}{\epsilon_K^2} & -\frac{\beta_3}{\epsilon_K} \end{bmatrix} \end{aligned} \tag{31}$$

The characteristic equation of the linearized system is

$$\begin{aligned} \Delta(s) &= \det(sI - A_1) \\ &= s^3 + \frac{\beta_3}{\epsilon_K}s^2 + \frac{\beta_2}{\epsilon_K^2}K_{bb}s + \frac{\beta_1}{\epsilon_K^3}K_{bb} \end{aligned} \tag{32}$$

The necessary and sufficient conditions for stability are obtained by using the Routh-Hurwitz criterion. Therefore, we get $\beta_1, \beta_2, \beta_3 > 0$ and $\epsilon_K > 0$. These conditions guarantee that A_1 is a stable matrix.

5. SIMULATION RESULTS

The values of the parameters of the ball and beam system are listed in Table 1.

Parameter	Description	Value
L_{beam}	Beam length	42.55cm
r_{arm}	Distance between SRV02 output gear	2.54cm
r_b	Radius of ball	1.27cm
m_b	Mass of ball	0.064kg
g	Gravitational constant	9.81 m/s^2
J_b	Moment of inertia of ball	4.1290$\times 10^{-6}$kg.m^2
K	Steady-state gain	1.76 rad/sv
τ	Time constant	0.0285s

Table 1 Values of the parameters of the ball and beam system.

For the nonlinear observer, we select L = [-8, -24, -32, -16]^T. The controller parameters β_1,β_2 and β_3 are selected such that the roots of the polynomial Δ are $-1 \pm i$, -10, and $\Gamma = 100$. It can be concluded that the responses of the ball position is fast and input V_m is within reasonable magnitude. Figure 2 shows the simulation results demonstrate the effects of gain-scaling factor ϵ_L and ϵ_K. As expected, we find that the proposed control schemes work well by scaling factor ϵ_L and ϵ_K. Figure 1 shows the simulation results when we select $\epsilon_L = 0.5$, $\epsilon_K = 0.8$.

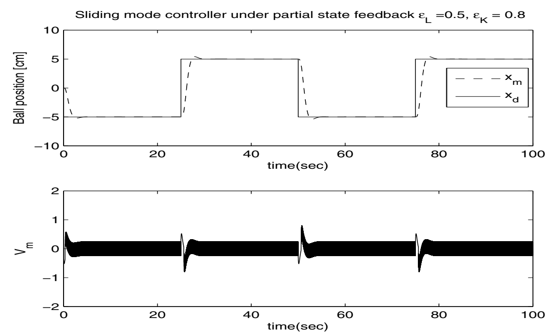

Fig. 1 Simulation results : $\epsilon_L = 0.5$, $\epsilon_K = 0.8$

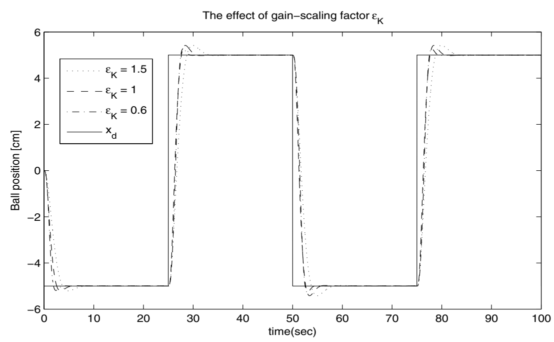

Fig. 2 Simulation results : The effect of gain-scaling factor ϵ_L and ϵ_K.

6. EXPERIMENTAL RESULTS

The experiment is carried out on the Quanser's ball and beam system. The composition of equipment is shown Fig 2. Also, the power module used is the Quanser

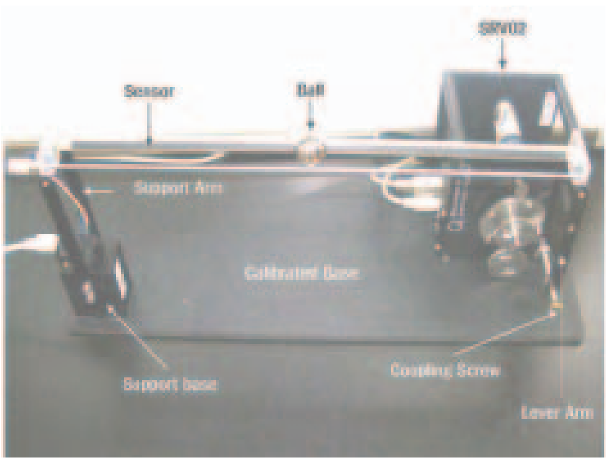

Fig. 3 Ball and beam system

UPM1503 with $\pm 15V$ and $3A$ output. Figure 4 shows the ball position and input V_m. Notice that there is high overshoot and slow convergence in the ball position response x_d. However, figure 5 shows better results than figure 4. Thus, we find that The system performance is properly adjusted by ϵ_L and ϵ_K.

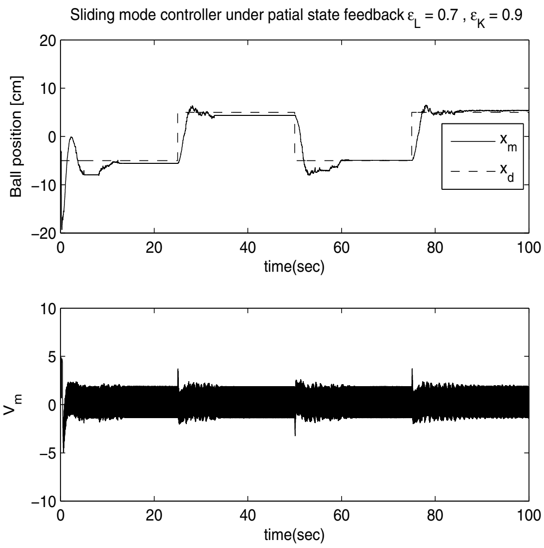

Fig. 4 Experimental results : $\epsilon_L = 0.7$, $\epsilon_K = 0.9$

7. CONCLUSION

In this paper, we investigated the sliding mode control under partial state feedback for ball and beam system. We show that stability analysis and convergence analysis of the observer. Moreover, it is shown that the system response can be adjusted by using ϵ_L and ϵ_K

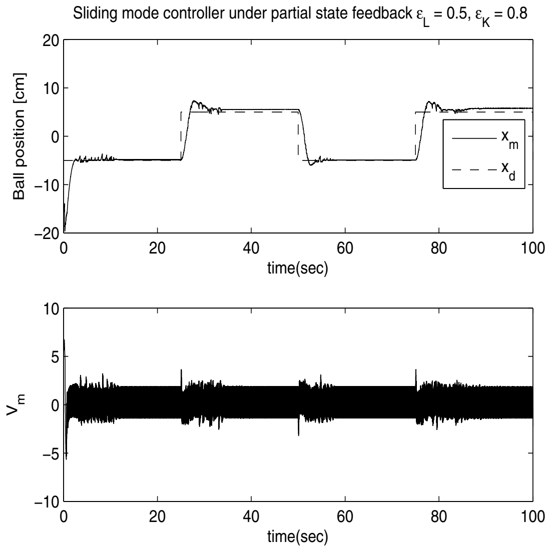

Fig. 5 Experimental results : $\epsilon_L = 0.5$, $\epsilon_K = 0.8$

8. ACKNOWLEDGMENT

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2009-0063942).

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