Robust Stabilization of Normalized Coprime Factor Plant Descriptions with H_∞-Bounded Uncertainty

KEITH GLOVER, MEMBER, IEEE, AND DUNCAN MCFARLANE, MEMBER, IEEE

***Abstract*—The problem of robustly stabilizing a family of linear systems is explicitly solved in the case where the family is characterized by H_∞ bounded perturbations to the numerator $\tilde{N}$ and denominator $\tilde{M}$ of the normalized left coprime factorization of a nominal system. This problem can be reduced to a Nehari extension problem directly and gives an optimal stability margin $\sqrt{1-\|[\tilde{N},\ \tilde{M}]\|_H^2}$. All controllers satisfying a suboptimal stability margin are characterized and explicit state-space formulas are given.**

I. Introduction

THE problem of robust stability of closed-loop systems has received a considerable amount of attention in recent years. In particular, the H_∞ approach to optimal control system analysis and design has provided some promising results in the area of robust stabilization of plants with *unstructured uncertainties*. Unstructured uncertainty in a process is uncertainty about which there is no information available except that an upper-bound on its magnitude, as a function of frequency, can be estimated.

In optimal H_∞ design, it is necessary to model plant uncertainty as a separate transfer function from the nominal plant model, and two common approaches are to model the uncertainty in a *multiplicative* or *additive* way with respect to the nominal plant. These forms of uncertainty are investigated, for example, in Doyle and Stein [6] and Chen and Desoer [3], and necessary and sufficient conditions are established for a given controller to stabilize all such perturbed plants. The robust stability condition in [3] gives a test on the H_∞-norm of a certain closed-loop transfer function, and hence the existence of a robustly stabilizing controller can be determined via H_∞ optimization techniques originated by Zames [37]. (See the monograph by Francis [8] and the references therein.) The above observation for robust stabilization under unstructured perturbations was made by Kimura [18] for the SISO case, with multivariable extensions in Vidyasagar and Kimura [35], Glover [10], and Verma *et al.* [30].

An alternative expression for plant uncertainty has been advocated by Vidyasagar in [31]–[33] in terms of additive stable perturbations to the factors in a coprime factorization of the plant. He shows that such a family of perturbations is particularly appropriate for feedback system analysis. The present paper also considers this class of unstructured perturbation (which is assumed to have bounded H_∞-norm) and obtains a surprisingly explicit and intuitively appealing solution to the corresponding robust stabilization problem when the coprime factorization is *normalized*. In particular, it will be shown that if the plant transfer function is written with a coprime factorization

$$G=\tilde{M}^{-1}\tilde{N}$$

where $\tilde{M}$, $\tilde{N}$ are *normalized* such that

$$\tilde{M}\tilde{M}^*+\tilde{N}\tilde{N}^*=I$$

then the family of plants

$$\mathcal{G}_\epsilon=\{(\tilde{M}+\Delta_M)^{-1}(\tilde{N}+\Delta_N) : \|[\Delta_M,\ \Delta_N]\|_\infty<\epsilon\}$$

can be stabilized by a single linear time-invariant controller if and only if

$$\epsilon^2\le 1-\|[\tilde{M},\ \tilde{N}]\|_H^2.$$

The problem of finding a maximum ϵ can be solved via standard H_∞ optimization techniques (see [8]). However, it will be shown here that all optimal controllers maximizing ϵ can be obtained from a Nehari extension problem: namely, if $\begin{bmatrix} U \\ V\end{bmatrix}$ is an optimal Nehari extension of the matrix function $\begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^*\end{bmatrix}$ satisfying

$$\left\|\begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^*\end{bmatrix}+\begin{bmatrix} U \\ V\end{bmatrix}\right\|_\infty=\|[\tilde{N},\ \tilde{M}]\|_H$$

then $K=UV^{-1}$ is a stabilizing controller achieving the maximum allowable stability margin, and $(U,\ V)$ is a right coprime factorization of K.

Section II gives some preliminary definitions, and some common results on coprime factors; Section III formulates the problem in the H_∞ framework; Section IV then demonstrates that all optimal controllers can be obtained via the Nehari extension approach. In addition, the *suboptimal* problem is introduced (that of obtaining all controllers which robustly stabilize a plant with a prespecified uncertainty level) and a parameterization of all such controllers is given. Section V gives explicit state-space formulas for all suboptimal controllers. Finally, concluding remarks are given in Section VI.

II. Preliminaries

A. Nomenclature and Definitions

All systems will be assumed linear, finite-dimensional, and time-invariant. The following notation is used. Rational matrix transfer functions are denoted $G(s)$ or G to differentiate from constant matrices. RL_∞ denotes the space of proper, real-rational functions with no pole on $s=j\omega$ with norm denoted $\|\cdot\|_\infty$. RH_∞ denotes the subspace of RL_∞ with no poles in the closed right-half plane. A^* is the complex conjugate transpose of A, and for real-rational functions of s, G^* denotes $[G(-\bar{s})]^*$. $\{\lambda_i(A),\ 1\le i\le n\}$ denotes the set of eigenvalues of A. A linear fractional transformation (LFT) will be denoted

$$\mathcal{F}_L\left(\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22}\end{bmatrix},K\right)\triangleq P_{11}+P_{12}K(I-P_{22}K)^{-1}P_{21}. \tag{2.1a}$$

Alternatively, if P_{21}^{-1} exists, an LFT can also be denoted

$$\mathfrak{J}_U[K]=(U_{11}K+U_{12})(U_{21}K+U_{22})^{-1} \tag{2.1b}$$

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K. Glover is with the Department of Engineering, University of Cambridge, Cambridge, CB2 1PZ England.
D. McFarlane is with the Broken Hill Proprietary Co., Ltd., Melbourne Research Laboratories, Clayton, Australia.

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where

$$U=\left[\begin{array}{c:c} U_{11} & U_{12} \\ \hdashline U_{21} & U_{22} \end{array}\right].$$

A state-space system is denoted $G \equiv (A, B, C, D)$ or

$$G \equiv \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{2.2}$$

where $G(s) = C(sI - A)^{-1}B + D$. If $G \in RH_\infty$ with a state-space realization as in (2.2), then the controllability and observability Gramians, P and Q, respectively, are defined as the solutions to the following Lyapunov equations:

$$AP+PA^*+BB^*=0 \tag{2.3}$$

$$A^*Q+QA+C^*C=0. \tag{2.4}$$

The Hankel singular values of G are defined $\{\sigma_i \triangleq \lambda_i^{1/2}(PQ),\ 1 \leq i \leq n\}$ and the Hankel norm, denoted $\|\cdot\|_H$ is the largest of these. A transfer is called *all-pass* if it is square and $G^*G = I$, *inner* if it is stable and $G^*G = I$, and *coinner* if it is stable and $GG^* = I$.

B. Coprime Factorization

Some results on coprime factorizations are now given. Only *left* coprime factor descriptions are stated here, as similar results for the *right* coprime factor case can be obtained by duality.

Definition 2.1: Matrices $\tilde{M}, \tilde{N} \in RH_\infty$ constitute a left coprime factorization (LCF) of G if and only if

a) $\tilde{M}$ is square, and $\det(\tilde{M}) \neq 0$

b) $G=\tilde{M}^{-1}\tilde{N}$ (2.5)

c) there exists $V, U \in RH_\infty$ such that

$$\tilde{M}V+\tilde{N}U=I. \tag{2.6}$$

■

An arbitrarily large number of LCF's can be generated for a single plant G (see Vidyasagar [33, Theorem 4.43]). A particular left coprime factorization of G is one in which the factors $\tilde{N}, \tilde{M}$ are *normalized.*

Definition 2.2: A left coprime factorization of G as defined above is normalized if and only if

$$\tilde{N}\tilde{N}^*+\tilde{M}\tilde{M}^*=I \quad \text{for all } s \tag{2.7}$$

or equivalently, if and only if the matrix $[\tilde{N}, \tilde{M}]$ is coinner. ■

A state-space construction for the normalized left (respectively, right) coprime factorizations can be obtained in terms of the solution to the generalized control (respectively, filter) algebraic Riccati equation.

Generalized Control Algebraic Riccati Equation (GCARE):

$$(A-BS^{-1}D^*C)^*X+X(A-BS^{-1}D^*C) - XBS^{-1}B^*X+C^*(I-DS^{-1}D^*)C=0 \tag{2.8}$$

where $S \triangleq I + D^*D$; and

Generalized Filter Algebraic Riccati Equation (GFARE):

$$(A-BD^*R^{-1}C)Z+Z(A-BD^*R^{-1}C)^* - ZC^*R^{-1}CZ+B(I-D^*R^{-1}D)B^*=0 \tag{2.9}$$

where $R \triangleq I + DD^*$. Some results on solutions to GCARE and GFARE are given in Appendix A.

Meyer and Franklin [25] state the state-space construction for normalized right coprime factors and this has been extended by Vidyasagar [34] to the case where G is not necessarily strictly proper. The equivalent normalized LCF construction is now given.

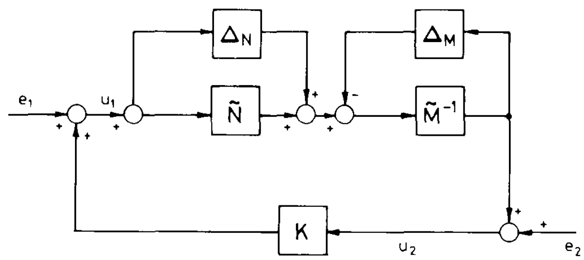

Fig. 1. Left coprime factor perturbations.

Lemma 2.1: Let $G(s) = C(sI - A)^{-1}B + D$ with (A, B, C) minimal. If

$$H=-(ZC^*+BD^*)R^{-1}$$

where Z is the unique, positive definite solution to GFARE, then

$$\tilde{N} \triangleq R^{-1/2}C(sI-A-HC)^{-1}(B+HD)+R^{-1/2}D \tag{2.10a}$$

$$\tilde{M} \triangleq R^{-1/2}+R^{-1/2}C(sI-A-HC)^{-1}H \tag{2.10b}$$

is a normalized LCF of G such that $G = \tilde{M}^{-1}\tilde{N}$. ■

III. Problem Formulation

A particular robust stabilization problem is now going to be considered which uses the *normalized* left coprime factorization representation of the nominal plant $G(s)$. (In Section IV it is shown that this choice of coprime factorization has a number of advantages over other coprime factorizations.)

Let the nominal plant model have a normalized left coprime factorization $\tilde{N}, \tilde{M}$ such that

$$G=\tilde{M}^{-1}\tilde{N}. \tag{3.1}$$

Then any perturbed plant can be written

$$G_\Delta=(\tilde{M}+\Delta_M)^{-1}(\tilde{N}+\Delta_N) \tag{3.2}$$

where Δ_N, Δ_M are stable unknown transfer functions which represent the uncertainty in the nominal plant model.

The robust design objective is to stabilize not only the nominal plant G, but the family of perturbed plants defined by

$$\mathcal{G}_\epsilon=\{(\tilde{M}+\Delta_M)^{-1}(\tilde{N}+\Delta_N) : \|[\Delta_M, \Delta_N]\|_\infty<\epsilon\} \tag{3.3}$$

using a feedback controller K (see Fig. 1).

It is demonstrated by Vidyasagar [31]–[33] that this description of plant uncertainty has a number of advantages over additive or multiplicative unstructured uncertainty models. For example, the number of unstable poles may change as the plant is perturbed.

Some preliminary definitions will now be given for this particular problem.

Definition 3.1: The feedback system of Fig. 1 (with $\Delta_M = \Delta_N = 0$) will be denoted (G, K) and called internally stable if and only if

a) $(I-GK)^{-1}, K(I-GK)^{-1}, (I-GK)^{-1}G,$
$(I-KG)^{-1} \in RH_\infty$

b) $\det(I-GK)(\infty)\neq 0.$ ■

Definition 3.2: The feedback system of Fig. 1, denoted $(\tilde{M}, \tilde{N}, K, \epsilon)$ is robustly stable if and only if (G_Δ, K) is internally stable for all $G_\Delta \in \mathcal{G}_\epsilon$. ■

The maximum value of ϵ while retaining stability is called the *stability margin* for this problem. Hence, ϵ is a limitation on the size of perturbation that can exist without destabilizing the closed-loop system of Fig. 1.

Further, if there exists K such that $(\tilde{M}, \tilde{N}, K, \epsilon)$ is robustly stable, then $(\tilde{M}, \tilde{N}, \epsilon)$ is said to be *robustly stabilizable* with stability margin ϵ.

Necessary and sufficient conditions for robust stability [3] will now be stated, and then it will be shown that this problem fits neatly into the standard H_∞ framework.

Lemma 3.1: The feedback system $(\tilde{M}, \tilde{N}, K, \epsilon)$ is robustly stable if and only if (G, K) is internally stable and

$$\left\| \begin{bmatrix} K(I-GK)^{-1}\tilde{M}^{-1} \\ (I-GK)^{-1}\tilde{M}^{-1} \end{bmatrix} \right\|_\infty \le \epsilon^{-1}. \tag{3.4}$$

Equivalently, $(\tilde{M}, \tilde{N}, \epsilon)$ is robustly stabilizable if and only if

$$\inf_K \left\| \begin{bmatrix} K(I-GK)^{-1}\tilde{M}^{-1} \\ (I-GK)^{-1}\tilde{M}^{-1} \end{bmatrix} \right\|_\infty \le \epsilon^{-1} \tag{3.5}$$

where the infimum is chosen over all stabilizing controllers K. ■

Equation (3.5) is in the form of an H_∞ optimization problem, which allows ϵ^{-1} to be chosen as small as possible.

The problem stated above can be converted to the general formulation of Doyle [5]. Let

$$P \triangleq \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right] = \left[\begin{array}{c|c} 0 & I \\ \tilde{M}^{-1} & G \\ \hline \tilde{M}^{-1} & G \end{array}\right]. \tag{3.6}$$

Then (3.5) is equivalent to

$$\inf_K \|\mathfrak{F}_L(P, K)\|_\infty \le \epsilon^{-1} \tag{3.7}$$

where K is chosen over all stabilizing controllers. (P is referred to as the H_∞ standard plant.)

The normalized coprime factor robust stabilization problem is now in the standard form for an H_∞ optimization problem. The next section gives an explicit solution to this particular problem.

Remark 3.1: It should be noted that the problem as stated so far could be applied to *any* left coprime factorization of G. The H_∞ optimization problem in (3.5) was formulated in Vidyasagar and Kimura [35] for any left coprime factorization, and could be solved using the standard iterative procedures outlined, for example, in Francis [8], Chu *et al.* [4], or Foo and Postlethwaite [7]. Sections IV and V show that an advantage of selecting the *normalized* coprime factorization is that the problem can be solved *exactly* in a remarkably simple way and that the computationally expensive iterative procedure can be avoided.

IV. Characterizing All Solutions

A. Solution Via a Nehari Extension Approach

In Section III, the H_∞ standard plant was formulated for the normalized LCF robust stabilization problem to which the standard solution procedure of [8] can be applied. The results presented in this section, however, present a diversion from these standard solution methods. It will be shown that the coprime factors of the controller can be generated *directly* from the normalized coprime factors of the plant by obtaining a *Nehari extension* of the matrix transfer function $\begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix}$. It is hoped that this surprisingly explicit result will provide new insight into the H_∞ optimization procedure and its links with Nehari and Hankel norm approximation problems. In addition, it will be shown that, for this particular problem, the maximum stability margin can be calculated exactly, in a simple way.

The main results of this section will now be stated, which relate the solution of the LCF robust stabilization problem to the solution of a particular Nehari extension problem, and in addition, the optimal stability margin for this problem is explicitly stated.

Theorem 4.1: A controller K is stabilizing and satisfies

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I-GK)^{-1}\tilde{M}^{-1} \right\|_\infty \le \gamma \tag{4.1}$$

if and only if K has an RCF: $K = UV^{-1}$ for some $U, V \in RH_\infty$ satisfying

$$\left\| \begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix} + \begin{bmatrix} U \\ V \end{bmatrix} \right\|_\infty \le (1-\gamma^{-2})^{1/2}. \tag{4.2}$$

Theorem 4.2:

a) Optimal solutions to the normalized LCF robust stabilization problem give

$$\inf_K \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I-GK)^{-1}\tilde{M}^{-1} \right\|_\infty = \{1 - \|[\tilde{N}, \tilde{M}]\|_H^2\}^{-1/2} \tag{4.3}$$

where the infimum is taken over all stabilizing K.

b) The maximum robust stability margin is

$$\epsilon_{\max} = \{1 - \|[\tilde{N}, \tilde{M}]\|_H^2\}^{1/2} > 0. \tag{4.4}$$

c) All optimal controllers are given by $K = UV^{-1}$ where U, V, $\in RH_\infty$ satisfy

$$\left\| \begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix} + \begin{bmatrix} U \\ V \end{bmatrix} \right\|_\infty = \|[\tilde{N}, \tilde{M}]\|_H. \tag{4.5}$$

The proof of Theorem 4.1 will need the following lemma. (The proof is given in Appendix C.)

Lemma 4.1:

a) Let $E_1, E_2 \in RL_\infty$ satisfy

$$\left\| \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \right\|_\infty \le \alpha < 1. \tag{4.6}$$

Then, $E_1(I - E_2)^{-1} \in RL_\infty$ and

$$\|E_1(I-E_2)^{-1}\|_\infty \le \alpha(1-\alpha^2)^{-1/2}. \tag{4.7}$$

b) For any $F \in RH_\infty$ satisfying $\|F\|_\infty \le \alpha(1 - \alpha^2)^{-1/2}$, there exists $E_1, E_2 \in RH_\infty$ such that $E_1(I - E_2)^{-1} = F$ and (4.6) is satisfied.

Proof of Theorem 4.1:

a) Necessity: Suppose K is a stabilizing controller satisfying (4.1) and with RCF $(\hat{U}, \hat{V})$, then an equivalent RCF is (U, V) with

$$\begin{bmatrix} U \\ V \end{bmatrix} = -\gamma^{-2} \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} (\tilde{M}\hat{V} - \tilde{N}\hat{U})^{-1} \tag{4.8}$$

since $(\tilde{M}\hat{V} - \tilde{N}\hat{U})^{-1} \in RH_\infty$ by internal stability requirements.

Further, substituting for K and G in the closed loop gives

$$\begin{bmatrix} K \\ I \end{bmatrix} (I-GK)^{-1}\tilde{M}^{-1} = \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} (\tilde{M}\hat{V} - \tilde{N}\hat{U})^{-1} = -\gamma^2 \begin{bmatrix} U \\ V \end{bmatrix}$$

and as $\begin{bmatrix} M^* & N^* \\ -\tilde{N} & \tilde{M} \end{bmatrix}$ is all pass (where (N, M) is a *normalized* RCF of G satisfying $N^*N + M^*M = I$), we can exploit the unitary invariance of the H_∞ norm to obtain

$$\gamma^2 \geq \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I-GK)^{-1}\tilde{M}^{-1} \right\|_\infty^2$$
$$= \left\| \begin{bmatrix} M^* & N^* \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} (\tilde{M}\hat{V}-\tilde{N}\hat{U})^{-1} \right\|_\infty^2$$
$$= \left\| \begin{bmatrix} (M^*\hat{U}+N^*\hat{V})(\tilde{M}\hat{V}-\tilde{N}\hat{U})^{-1} \\ I \end{bmatrix} \right\|_\infty^2$$
$$= 1 + \|(M^*\hat{U}+N^*\hat{V})(\tilde{M}\hat{V}-\tilde{N}\hat{U})^{-1}\|_\infty^2. \quad (4.9)$$

Now consider

$$\left\| \begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix} + \begin{bmatrix} U \\ V \end{bmatrix} \right\|_\infty^2$$
$$= \left\| \begin{bmatrix} M^* & N^* \\ -\tilde{N} & \tilde{M} \end{bmatrix} \left\{ \begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix} + \begin{bmatrix} U \\ V \end{bmatrix} \right\} \right\|_\infty^2$$
$$= \left\| \begin{bmatrix} -\gamma^{-2}(M^*\hat{U}+N^*\hat{V})(\tilde{M}\hat{V}-\tilde{N}\hat{U})^{-1} \\ (1-\gamma^{-2})I \end{bmatrix} \right\|_\infty^2 \quad \text{by (4.8)}$$
$$\leq (1-\gamma^{-2})^2 + \gamma^{-4}(\gamma^2-1) \quad \text{from (4.9)}$$
$$= 1-\gamma^{-2}$$

and necessity is proven.

b) Sufficiency: Assume now that $U, V \in RH_\infty$ satisfies (4.2) and define

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} M^* & N^* \\ -\tilde{N} & \tilde{M} \end{bmatrix} \left\{ \begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix} + \begin{bmatrix} U \\ V \end{bmatrix} \right\}$$
$$= \begin{bmatrix} M^*U+N^*V \\ I+(\tilde{M}V-\tilde{N}U) \end{bmatrix}. \quad (4.10)$$

Then $\|[\begin{smallmatrix} E_1 \\ E_2 \end{smallmatrix}]\|_\infty^2 \leq 1-\gamma^{-2}$ by assumption. Further, for $K = UV^{-1}$

$$\begin{bmatrix} M^* & N^* \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} K \\ I \end{bmatrix} (I-GK)^{-1}\tilde{M}^{-1}$$
$$= \begin{bmatrix} M^* & N^* \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} (\tilde{M}V-\tilde{N}U)^{-1}$$
$$= \begin{bmatrix} E_1(E_2-I)^{-1} \\ I \end{bmatrix}$$

and applying Lemma 4.1

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I-GK)^{-1}\tilde{M}^{-1} \right\|_\infty^2 \leq 1 + \frac{(1-\gamma^{-2})}{\gamma^{-2}} = \gamma^2.$$

Finally, K will be stabilizing (and U, V is an RCF of K) since $(\tilde{M}V - \tilde{N}U)^{-1} = (E_2 - I)^{-1} \in RH_\infty$ by the small gain theorem since $\|E_2\|_\infty < 1$ by assumption and $E_2 \in RH_\infty$. ∎

Proof of Theorem 4.2: This is an immediate consequence of Theorem 4.1 and Nehari's theorem, and the following lemma. (The proof is given in Appendix C.)

Lemma 4.2: A normalized left coprime factorization $(\tilde{N}, \tilde{M})$ satisfies

$$\|[\tilde{N}, \tilde{M}]\|_H < 1.$$ ∎

Hence, $1 - \gamma^{-2} \geq \|[\tilde{N}, \tilde{M}]\|_H^2$, with equality achievable by choosing $[\begin{smallmatrix} U \\ V \end{smallmatrix}]$ to be an optimal Nehari extension of $[\begin{smallmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{smallmatrix}]$. Lemma 4.2 guarantees that $\epsilon_{max} > 0$. ∎

Remark 4.1: The main implication of Theorems 4.1 and 4.2 is that all H_∞ optimal controllers for this problem can be obtained by solving a related Nehari extension problem. Theoretically, this is a more straightforward approach than a general H_∞ approach, and it is hoped that this will give greater insight into the nature of this problem.

Remark 4.2: The maximum stability margin given in (4.4) is surprisingly explicit. The H_∞ optimization problem posed in (3.5) is a "two-block" problem for which an iterative solution is normally required to approximate ϵ_{max}. It is possible to show [13] that (4.4) can be obtained in an alternative way by a simplification of the approach of [8].

Remark 4.3: It is possible to obtain a dual result for the maximum stability margin if a normalized right coprime factorization (N, M) of the nominal plant is used. For any nominal plant, the optimal stability margin for the normalized left and right coprime factor problems is the same. It is also obvious that all optimal controllers for the normalized RCF problem can be obtained from $\tilde{U}, \tilde{V} \in RH_\infty$ such that $K = \tilde{V}^{-1}\tilde{U}$, where $[\tilde{U}, \tilde{V}]$ is the optimal Nehari extension of $[-N^*, M^*]$. The choice between the two approaches in design is related to particular robust stability and performance objectives, and this is discussed elsewhere [22].

Remark 4.4: In the robust stabilization of $(G + \Delta)$ with $\|\Delta\|_\infty < \beta$ it has been observed that the largest robustly stabilizable region with a single controller has a nonstabilizable plant on its boundary [10]. This result has been exploited by Khargonekar *et al.* [17] to show, for example, that nonlinear and time-varying controllers can do no better. The same comments apply to unstructured multiplicative perturbations. However, in the case of coprime factor uncertainty, it can be shown that, in general, the *nearest* unstabilizable system to a nominal $[\tilde{N}, \tilde{M}]$ (in the H_∞ norm) is beyond the boundary of the largest robustly stabilizable set, where by "nearest unstabilizable system," we mean here the nearest $G_\Delta = (N + \Delta_N)(M + \Delta_M)^{-1}$ such that the pair $(N + \Delta_N, M + \Delta_M)$ is no longer coprime. That is, there is RHP pole/zero cancellation.

Consider the following second-order example. A nominal plant $G = 12/s(s + 5)$ has a normalized left coprime factorization given by

$$[\tilde{N}, \tilde{M}] = [12, s(s+5)]/(s+3)(s+4).$$

Routine calculations yield that $\|[\tilde{N}, \tilde{M}]\|_H = 0.79156$, and hence $\epsilon_{max} = 0.61109$. However, in the single-input, single-output case, the distance to the nearest stabilizable plant is given by

$$d = \inf_{\text{Re}(s)>0} (|\tilde{N}|^2 + |\tilde{M}|^2)^{1/2}$$

and a search on Re $(s) > 0$ indicates that $d = 0.6146$, and thus the nearest unstabilizable system to $[\tilde{N}, \tilde{M}]$ in this norm is very slightly beyond the boundary of the largest robustly stabilizable set. Hence, the above remarks concerning nonlinear and time-varying controllers do not necessarily apply in this case.

B. Parametrizing All Controllers

A related problem to the optimal H_∞ problem posed in Section III is the so-called *suboptimal* problem of obtaining the set of stabilizing controllers K such that

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I-GK)^{-1}\tilde{M}^{-1} \right\|_\infty \leq \gamma \quad (4.11)$$

where $\gamma (> \epsilon_{max}^{-1})$ is some prespecified tolerance level for the allowable uncertainty. Theorem 4.1 shows that this is equivalent to finding all stable extensions of $[\begin{smallmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{smallmatrix}]$, designated Q, such that

$$\left\| \begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix} + Q \right\|_\infty \leq \alpha \quad (4.12)$$

where $\alpha = (1 - \gamma^{-2})^{1/2}$.

Appendix B gives a characterization of all such Q, and it is thus possible to characterize all controllers achieving (4.11).

Theorem 4.3: For $1 > \alpha > \| [\tilde{N}, \tilde{M}] \|_H$, let the parametrization of all extensions of $\begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix}$ such that $Q \in RH_\infty$ and

$$\left\| \begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix} + Q \right\|_\infty \le \alpha$$

be given by (see Appendix B)

$$Q = \mathfrak{F}_J[\Phi] \tag{4.13}$$

$\Phi \in RH_\infty^{(p+m)\times p}$, $\|\Phi\|_\infty \le 1$, and $J \in RH_\infty^{((p+m)+p)\times(m+p+p)}$

$$J = \left[\begin{array}{cc|c} J_{11} & -\alpha J_{12} & J_{12} \\ \hline J_{21} & I - \alpha J_{22} & J_{22} \end{array} \right]. \tag{4.14}$$

Then, all stabilizing controllers satisfying (4.11) are given by

$$K = (\tilde{J}_{11u}\tilde{\Phi} + \tilde{J}_{12u})(\tilde{J}_{11v}\tilde{\Phi} + \tilde{J}_{12v})^{-1} \tag{4.15}$$

where $J_{11} = \begin{bmatrix} J_{11u} \\ J_{11v} \end{bmatrix}$, $J_{12} = \begin{bmatrix} J_{12u} \\ J_{12v} \end{bmatrix}$ are partitioned conformally with $\begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix}$ and

$$\begin{bmatrix} \tilde{J}_{11u} & \tilde{J}_{12u} \\ \tilde{J}_{11v} & \tilde{J}_{12v} \end{bmatrix} \triangleq \begin{bmatrix} \gamma J_{11u} & J_{12u} \\ \gamma J_{11v} & J_{12v} \end{bmatrix} \tag{4.16}$$

and $\tilde{\Phi} \in RH_\infty^{m\times p}$ such that $\|\tilde{\Phi}\|_\infty \le 1$.

Proof: From (4.14), all Q are given by

$$Q = (J_{11}\Phi_1 - \alpha J_{12}\Phi_2 + J_{12})(J_{21}\Phi_1 + (I - \alpha J_{22})\Phi_2 + J_{22})^{-1} \tag{4.17}$$

where $\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$, and $\Phi_1 \in RH_\infty^{m\times p}$, $\Phi_2 \in RH_\infty^{p\times p}$. Noting that as $Q \in RH_\infty$ and $J \in RH_\infty$ that $(J_{21}\Phi_1 + (I - \alpha J_{22})\Phi_2 + J_{22})$ is a unit in RH_∞, then from basic coprime factorization theory (see [33]) all coprime factors of the controller (given by Q) are also given by

$$\begin{aligned} Q' &= (J_{11}\Phi_1 - \alpha J_{12}\Phi_2 + J_{12}) \\ &= J_{11}\Phi_1 + J_{12}(I - \alpha\Phi_2) \end{aligned} \tag{4.18}$$

where $(I - \alpha\Phi_2) \in RH_\infty$, and also $(I - \alpha\Phi_2)^{-1} \in RH_\infty$ by the small gain theorem. Hence, without loss of generality, coprime factorizations of all controllers are given by

$$Q'' = J_{11}\Phi_1(I - \alpha\Phi_2)^{-1} + J_{12}.$$

Next, noting that

$$\left\| \begin{bmatrix} \alpha\Phi_1 \\ \alpha\Phi_2 \end{bmatrix} \right\|_\infty \le \alpha < 1 \tag{4.19}$$

then by Lemma 4.1 a)

$$\begin{aligned} &\Rightarrow \|\alpha\Phi_1(I - \alpha\Phi_2)^{-1}\|_\infty \le \alpha(1 - \alpha^2)^{-1/2} \\ &\Rightarrow \|\tilde{\Phi}_1\|_\infty \le (1 - \alpha^2)^{-1/2} \end{aligned} \tag{4.20}$$

where $\tilde{\Phi}_1 \triangleq \Phi_1(I - \alpha\Phi_2)^{-1} \in RH_\infty$. Conversely, by Lemma 4.1 b), for any $\tilde{\Phi}_1 \in RH_\infty$ satisfying (4.20) there exists Φ_1, $\Phi_2 \in RH_\infty$ satisfying (4.19).

Hence, coprime factors of all controllers are now given by

$$Q'' = J_{11}\tilde{\Phi}_1 + J_{12} \tag{4.21}$$

where $\tilde{\Phi}_1 \in RH_\infty^{m\times p}$ and $\|\tilde{\Phi}_1\|_\infty \le (1 - \alpha^2)^{-1/2}$ $(= \gamma)$. Alternatively, by scaling $\tilde{\Phi}_1$ by γ^{-1} and J_{11} by γ

$$Q'' = \tilde{J}_{11}\tilde{\Phi} + \tilde{J}_{12} \tag{4.22}$$

where $\tilde{J}_{11} = \gamma J_{11}$, $\tilde{J}_{12} = J_{12}$, $\tilde{\Phi} \in RH_\infty^{m\times p}$, and $\|\tilde{\Phi}\|_\infty \le 1$.

Noting that $Q'' = \begin{bmatrix} U \\ V \end{bmatrix}$, where U, V are coprime factors of K: $K = UV^{-1}$, then appropriate partitioning of J_{11}, J_{12} as $\begin{bmatrix} J_{11u} \\ J_{11v} \end{bmatrix}$, $\begin{bmatrix} J_{12u} \\ J_{12v} \end{bmatrix}$ yields the desired result. ■

Remark 4.5: It is interesting to note that although there are $(p + m) \times p$ degrees of freedom associated with Q (via the arbitrary contraction Φ), there are only $m \times p$ degrees of freedom associated with the resulting suboptimal controller (via $\tilde{\Phi}$). The reason for this is that the suboptimal extension can generate right coprime factorizations by multiplying any one solution by a $p \times p$ unit in RH_∞. The remaining $m \times p$ degrees of freedom are associated with the controller (rather than the factorization) and would also be obtained had the controller been derived via the standard H_∞ methods [23].

In the next section, a state-space characterization of all suboptimal controllers will be given.

V. A State-Space Representation for All Suboptimal Controllers

A. Characterizing All Controllers

From Lemma 2.1, a minimal state-space representation of $\begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix}$ is

$$\left[\begin{array}{c|c} -(A^o)^* & C^*R^{-1/2} \\ \hline (B + HD)^* & -D^*R^{-1/2} \\ -H^* & R^{-1/2} \end{array} \right] \tag{5.1}$$

and solutions to the associated Lyapunov equations [see (2.3) and (2.4)] are given by

$$P = -X(I + ZX)^{-1} \qquad \text{by (A.18)} \tag{5.2}$$

and

$$Q = -Z \qquad \text{by (A.16)}. \tag{5.3}$$

The results of Theorem 4.3 combined with Lemma B.2 (Appendix B) yield the following state-space parametrization for all suboptimal controllers achieving tolerance level γ.

Lemma 5.1: All controllers for the normalized LCF robust stabilization problem satisfying $\|\mathfrak{F}(P, K)\|_\infty \le \gamma$, for $\gamma > (1 - \|[\tilde{N}, \tilde{M}]\|_H^2)^{-1/2}$, are given by

$$K = \mathfrak{F}_L[\Phi] \tag{5.4}$$

where L has the state-space form

$$L = \left[\begin{array}{c|c} L_{11} & L_{12} \\ \hline L_{21} & L_{22} \end{array} \right]$$

$$= \left[\begin{array}{c|c|c} A^c & -\gamma^2 W_1^{*-1} B S^{-1/2} & \gamma^2 \zeta^{-1} W_1^{*-1} Z C^* R^{-1/2} \\ \hline F & S^{-1/2} & \zeta^{-1} D^* R^{-1/2} \\ \hline C + DF & DS^{-1/2} & -\zeta^{-1} R^{-1/2} \end{array} \right] \tag{5.5}$$

where $\zeta = (\gamma^2 - 1)^{1/2}$,

$$W_1 \triangleq I + (XZ - \gamma^2 I), \tag{5.6}$$

$\Phi \in RH_\infty^{m\times p}$ with $\|\Phi\|_\infty \le 1$, and all other terms are as defined in Appendix A.

Proof: As $[-\tilde{N} \; \tilde{M}]$ is a coinner function, the suboptimal extension will have the form given in (B.6) in Appendix B. Noting from Theorem 4.3 that only the (1, 1) and (1, 2) blocks of this state-space expression are required to determine all controllers, the state-space form of L [as required for (5.4)] can immediately be written as

$$L = \left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array} \right]$$

and by noting that

$$D_\perp = \begin{bmatrix} S^{-1/2} \\ DS^{-1/2} \end{bmatrix}$$

is a suitable unitary completion of

$$\begin{bmatrix} D^*R^{-1/2} \\ -R^{-1/2} \end{bmatrix}$$

then

$$A_L = A^o \tag{5.7}$$

$$B_L = [W^{*-1}BS^{-1/2} \;\; \alpha^{-1}W^{*-1}QC^*R^{-1/2}] \tag{5.8}$$

$$C_L = \left[-\begin{pmatrix} (B+HD)^* \\ -H^* \end{pmatrix} P + \begin{pmatrix} D^*R^{-1/2} \\ -R^{-1/2} \end{pmatrix} R^{-1/2}C \right] \tag{5.9}$$

$$D_L = \begin{bmatrix} S^{-1/2} & \alpha^{-1}D^*R^{-1/2} \\ DS^{-1/2} & -\alpha^{-1}R^{-1/2} \end{bmatrix}. \tag{5.10}$$

In addition

$$W \triangleq PQ - \alpha^2 I$$

and P and Q are defined in (5.2) and (5.3).

Now, by noting that

$$W = XZ(I+XZ)^{-1} - \alpha^2 I$$

$$= (\gamma^{-2}(I+XZ) - I)(I+XZ)^{-1} \qquad \text{as } \alpha^2 = 1-\gamma^{-2}$$

$$\Rightarrow W^{*-1} = \gamma^2(I+ZX)W_1^{*-1} \tag{5.11}$$

where $W_1 \triangleq I + (XZ - \gamma^2 I)$, then, (5.8) can be written

$$B_L = \gamma^2(I+ZX)W_1^{*-1}[BS^{-1/2}, \; -\alpha^{-1}ZC^*R^{-1/2}]. \tag{5.12}$$

Next note that

$$H^*P - R^{-1}C = -H^*X(I+ZX)^{-1} - R^{-1}C$$

$$= R^{-1}(DB^*X - C)(I+ZX)^{-1} \qquad \text{by (A.10)}$$

$$= -(C+DF)(I+ZX)^{-1}$$

by (A.5), (A.6), and (A.9)

and

$$BP + D^*(H^*P - R^{-1}C)$$

$$= (B^*X + D^*(C+DF))(I+ZX)^{-1} \qquad \text{by (5.2)}$$

$$= -F(I+ZX)^{-1} \qquad \text{by (A.9).}$$

Hence, (5.9) can be written

$$C_L = \begin{bmatrix} -F \\ -(C+DF) \end{bmatrix} (I+ZX)^{-1}. \tag{5.13}$$

Finally, applying a state transformation of the form $-(I + ZX)^{-1}$, and by noting (A.13)

$$L = \left[\begin{array}{c|c} L_{11} & L_{12} \\ \hline L_{21} & L_{22} \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} A^c & -\gamma^2 W_1^{*-1}BS^{-1/2} & \gamma^2\alpha^{-1}W_1^{*-1}ZC^*R^{-1/2} \\ \hline F & S^{-1/2} & \alpha^{-1}D^*R^{-1/2} \\ \hline C+DF & DS^{-1/2} & -\alpha^{-1}R^{-1/2} \end{array}\right]$$

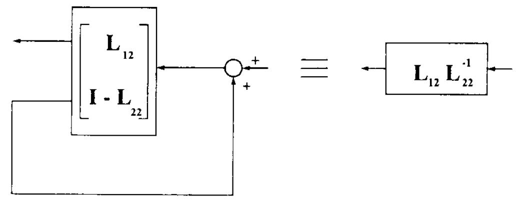

Fig. 2. Construction of K_o from coprime factors.

Scaling of L_{11}, L_{21}, as in (4.22), then noting that $\gamma^{-1}L$ also gives a parametrization of all suboptimal controllers ($\alpha^{-1} = \gamma(\gamma^2 - 1)^{-1/2}$) yields the required result given in (5.5). ■

B. The Central Controller

A particular controller is given by

$$K_o = L_{12}L_{22}^{-1} \tag{5.14}$$

which corresponds to $\Phi = 0$ which is the *central* or *maximum entropy* controller [27] for the selected tolerance level. The state-space formulas for K_o are now given.

Corollary 5.1: The central controller for tolerance level $\gamma > (1 - \|[\tilde{N}, \tilde{M}]\|_H^2)^{-1/2}$ has the state-space form

$$K_o = \left[\begin{array}{c|c} A^c + \gamma^2 W_1^{*-1}ZC^*(C+DF) & \gamma^2 W_1^{*-1}ZC^* \\ \hline B^*X & -D^* \end{array}\right] \tag{5.15}$$

where X (respectively, Z) solves GCARE (respectively, GFARE), and

$$F = -S^{-1}(D^*C + B^*X)$$

$$A^c = A + BF$$

$$W_1 = I + (XZ - \gamma^2 I).$$

Proof: Fig. 2 demonstrates that $K_o = L_{12}L_{22}^{-1}$ can be written as a unity feedback system, and the nth-order state-space form of K_o can be readily constructed from this and (5.5) without any state inflation. (The construction is straightforward and is omitted here.) ■

Remark 5.1: As has been recently stated by Glover and Doyle [9], the controller for the suboptimal H_∞ problem requires only the solution to two Riccati equations (as is required for an LQG controller). The present problem requires one Riccati equation to obtain $[\tilde{N}, \tilde{M}]$, and hence pose a standard H_∞ problem. Thereafter, one more Riccati equation need be solved as shown in Corollary 5.1. Further, noting from (5.2) and (5.3) that

$$\|[\tilde{N}, \tilde{M}]\|_H^2 = \lambda_{\max}(ZX(I+ZX)^{-1}) \tag{5.16}$$

then by (5.2), (5.3), and (4.4), it can be seen that (after some manipulation)

$$\gamma_{\min}^2 = \epsilon_{\max}^{-2} = 1 + \lambda_{\max}(ZX). \tag{5.17}$$

Hence, for the normalized LCF robust stabilization problem, the Riccati solutions are sufficient to specify the maximum achievable stability margin. Note that $\lambda_i(ZX)$ are precisely the closed-loop characteristic values defined in [15].

Remark 5.2: The H_∞ solution procedure proposed in [9] can be used to illustrate why the normalized LCF robust stabilization problem has an exact solution. This paper gives conditions for the existence of a stabilizing controller achieving tolerance level γ. In particular, the stabilizing solutions, X_∞ and Y_∞, to two Riccati equations must be positive semidefinite, and the spectral radius,

$\rho(X_\infty Y_\infty)$, be less than or equal to γ^2. To achieve an optimal solution, an iteration on γ yields the minimum tolerance such that these conditions hold. However, in the normalized LCF problem it can be shown [23] that

$$X_\infty = -\gamma^2 W_1^{-1} X = -\gamma^2 (I + XZ - \gamma^2 I)^{-1} X$$

$$Y_\infty = 0$$

where the matrices X, Z solve GCARE and GFARE, respectively, and γ is the required tolerance level. Hence, $Y_\infty \geq 0$ and $\rho(X_\infty Y_\infty) < \gamma^2$ for all γ, and

$$X_\infty \geq 0$$

$$\Leftrightarrow \gamma^2 \geq 1 + \lambda_{\max}(XZ).$$

We, therefore, have an explicit condition on γ, and no iteration is required.

Remark 5.3: The following manipulation demonstrates that the so-called two-block problem being considered in this paper is equivalent to a four-block problem; noting that the H_∞ norm is invariant under right multiplication by a coinner function, we have

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \tilde{M}^{-1} \right\|_\infty \leq \gamma$$

$$\Leftrightarrow \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} \tilde{M}^{-1} [\tilde{M}, \tilde{N}] \right\|_\infty \leq \gamma$$

$$\Leftrightarrow \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} [I, G] \right\|_\infty \leq \gamma.$$

Hence, the optimal controller for the normalized LCF robust stabilization problem is also the optimal controller for the problem: *find*

$$\inf_{K \text{ stabilizing}} \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} [I, G] \right\|_\infty .$$

Here, a combination of the closed-loop transfer function objectives $(I - GK)^{-1}$, $(I - GK)^{-1}G$, $K(I - GK)^{-1}$, and $K(I - GK)^{-1}G$ is being minimized. Using the procedure of [9] the solution to this four-block problem yields

$$X_\infty = \frac{\gamma^2}{\gamma^2 - 1} X$$

$$Y_\infty = Z$$

which are positive semidefinite for $\gamma > 1$, and

$$\rho(X_\infty Y_\infty) = \frac{\gamma^2}{\gamma^2 - 1} \rho(XZ) \leq \gamma^2$$

$$\Rightarrow \gamma^2 \geq 1 + \lambda_{\max}(XZ)$$

as before. (In [14], Grimble also shows the existence of an exact solution to a particular four-block problem.)

Remark 5.4: If the tolerance level is set to the minimum value specified in (5.17), a controller of degree $\leq n - 1$ is predicted [19]. For this, the *optimal* case, it can be shown [29] that as $\gamma \rightarrow \gamma_{\min}$ only $n - r$ states will exist in the realization in (5.15), where r is the multiplicity of $\lambda_{\max}(ZX)$. All such optimal controllers can be constructed from the optimal Nehari extension of $\left[\begin{smallmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{smallmatrix}\right]$(see Theorem 4.2) following the state-space construction in [12].

Remark 5.5: When $D \neq 0$, the feedback system is always well-posed since det $(I + DD^*) \neq 0$. However, if we now require that the system is well-posed in the face of infinitesimal time delays (see Willems [36]), the condition $\bar{\sigma}(D) < 1$ is sufficient.

VI. Conclusions

This paper has achieved the following:

1) introduced the idea of normalized coprime factors as a tool for obtaining robust stability using optimal H_∞ theory;

2) shown that the maximum stability margin in the normalized LCF robust stability problem can be simply and directly calculated;

3) demonstrated a link between robust stabilization using H_∞ optimization and Nehari extension problems and shown that the normalized LCF robust stabilization problem can be solved in this way;

4) given an explicit state-space characterization of all suboptimal controllers for the normalized LCF robust stabilization problem.

It has also been indicated that the theoretical simplifications allow a significant reduction in the computational effort required to obtain the H_∞ optimal controller for this problem since no iteration on γ is required.

Although only a particular H_∞ design approach has been considered here, it is claimed by the authors to be appropriate in a wide class of design problems. This is further discussed in McFarlane and Glover [22], where it will be shown that the normalized LCF robust stability problem can be incorporated into a systematic loop shaping design technique which considers performance as well as robust stability. A design example using this procedure is given in [21].

Appropriate methods for producing reduced-order controllers in this framework have been derived in [24] and are related to the work of Liu and Anderson [20], Anderson and Liu [1], and Meyer [26].

Appendix A

The Algebraic Riccati Equation

Consider the state-space model of the form

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \tag{A.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^r$ and A, B, C, D are time-invariant matrices of compatible sizes. This system is denoted (A, B, C, D) and in all further results, it is assumed to be a minimal realization of its transfer matrix

$$G(s) \triangleq C(sI - A)^{-1}B + D. \tag{A.2}$$

The two particular algebraic Riccati equations (ARE's) of interest in this work are the generalized control algebraic Riccati equation (GCARE)

$$(A - BS^{-1}D^*C)^*X + X(A - BS^{-1}D^*C) - XBS^{-1}B^*X + C^*R^{-1}C = 0 \tag{A.3}$$

and the generalized filtering algebraic Riccati equation (GFARE)

$$(A - BD^*R^{-1}C)Z + Z(A - BD^*R^{-1}C)^* - ZC^*R^{-1}CZ + BS^{-1}B^* = 0 \tag{A.4}$$

where

$$R \triangleq (I + DD^*) \tag{A.5}$$

$$S \triangleq (I + D^*D) \tag{A.6}$$

and by inspection $R^{-1} = I - DS^{-1}D^*$, $S^{-1} = I - D^*R^{-1}D$, $DS^{-1} = R^{-1}D$, and $DS = RD$.

Associated with these Riccati equations are the closed-loop control and filtering matrices, defined, respectively, as

$$A^c \triangleq A+BF \tag{A.7}$$

$$A^o \triangleq A+HC \tag{A.8}$$

where F, the control gain, and H, the filter gain, are defined

$$F \triangleq -S^{-1}(D^*C+B^*X) \tag{A.9}$$

$$H \triangleq -(BD^*+ZC^*)R^{-1}. \tag{A.10}$$

Noting that the controllability of $(A - BS^{-1}D^*C, BS^{-1/2})$ is uniquely implied by the controllability of (A, B) and that the observability of $(R^{-1/2}C, A - BD^*R^{-1}C)$ is uniquely implied by the observability of (C, A) (both can be shown by simple PBH tests), then the following theorems give sufficient (but not necessary) conditions for the existence and uniqueness of particular solutions to GCARE.

Theorem A.1 (Kalman [16]): If (A, B) is completely controllable, and (C, A) is completely observable, then there exists a unique solution, $X = X^* > 0$ to GCARE and the eigenvalues of A^c have strictly negative real parts. ∎

Remark A.1: It should be noted that considerably weaker conditions would be sufficient to yield the solutions stated in Theorems 1 and 2. The condition of minimality is assumed as this is compatible to assumptions made in the rest of the paper.

Remark A.2: Theorem 1 can be applied directly to GFARE and equivalent results obtained, if the system (A, B, C, D) is replaced by (A^*, C^*, B^*, D^*), X replaced by Z, and hence A^c replaced by A^o.

It can also be shown that X and Z defined in (A.3), (A.4), respectively, solve

$$(A-BS^{-1}D^*C)^*Z^{-1}+Z^{-1}(A-BS^{-1}D^*C) + Z^{-1}BS^{-1}B^*Z^{-1}-C^*R^{-1}C=0 \tag{A.11}$$

$$(A-BD^*R^{-1}C)X^{-1}+X^{-1}(A-BD^*R^{-1}C) + X^{-1}C^*R^{-1}CX^{-1}-BS^{-1}B^*=0. \tag{A.12}$$

It is also possible to relate the stabilizing solutions of GCARE and GFARE.

Theorem A.2 (Bucy [2]):

$$A^o=(I+ZX)A^c(I+ZX)^{-1} \tag{A.13}$$

$$(A^o)^*=(I+XZ)^{-1}(A^c)^*(I+XZ). \tag{A.14}$$

∎

(These were proven by Bucy for the case $D = 0$, but can easily be shown to apply to the $D \neq 0$ case as well.)

Finally for completeness, the stabilizing solutions of GCARE and GFARE can be shown to satisfy the following related Lyapunov equations:

$$XA^c+(A^c)^*X= -(C+DF)^*(C+DF)-F^*F = -C^*R^{-1}C-XBS^{-1}B^*X \tag{A.15}$$

$$A^oZ+Z(A^o)^*= -(B+HD)(B+HD)^*-HH^* = -BS^{-1}B^*-ZC^*R^{-1}CZ. \tag{A.16}$$

These are a direct result of (A.3) and (A.4). Further, two Lyapunov equations can be obtained by combining (A.3) with (A.14) and (A.4) with (A.14)

$$(Z^{-1}+X)^{-1}(A^c)^*+A^c(Z^{-1}+X)^{-1}= -BS^{-1}B^* \tag{A.17}$$

$$(X^{-1}+Z)^{-1}A^o+(A^o)^*(X^{-1}+Z)^{-1}= -C^*R^{-1}C. \tag{A.18}$$

Appendix B

Suboptimal Nehari Extensions

A state-space characterization will be derived here for all suboptimal extensions of an unstable function that is constrained to satisfy an inner requirement. We first state a more general result characterizing all suboptimal extensions of any unstable function. This is derived from [11].

Lemma B.1: All suboptimal extensions of a function $\boldsymbol{R}$, $\boldsymbol{R}^* \in RH_\infty^{m\times p}$, of degree n, with state-space form $\boldsymbol{R} = (A, B, C, D)$, given by

$$\|\boldsymbol{R}+\boldsymbol{Q}\|_\infty \le \alpha$$

can be written $\boldsymbol{Q} \in RH_\infty^{p\times m}$, where

$$\boldsymbol{Q}=\mathfrak{F}_V[\phi] \tag{B.1}$$

$\phi \in RH_\infty^{p\times m}$, $\|\phi\|_\infty \le 1$, and

$$V=\left[\begin{array}{c|c} V_{11} & V_{12} \\ \hline V_{21} & V_{22}\end{array}\right] = \left[\begin{array}{c|c:c} -A^* & W^{*-1}C^* & \alpha^{-1}W^{*-1}QB \\ \hline -(CP+DB^*) & I & -\alpha^{-1}D \\ \hdashline B^* & 0 & \alpha^{-1}I \end{array}\right] \tag{B.2}$$

where $-Q$ (respectively, $-P$) is the controllability (respectively, observability) Gramian of $\boldsymbol{R}^*$, and $W \triangleq (PQ - \alpha^2 I)$. ∎

We now characterize all suboptimal extensions of an unstable function $\boldsymbol{R}$ satisfying $\boldsymbol{R}^*\boldsymbol{R} = I$. (That is, $\boldsymbol{R}^*$ is coinner.)

Lemma B.2: Given a coinner function $\boldsymbol{R}^* \in RH_\infty^{m\times p}$, $m \le p$, of degree n, with $\boldsymbol{R}$ having state-space realization $\boldsymbol{R} = (A, B, C, D)$, then all transfer functions $\boldsymbol{Q} \in RH_\infty^{p\times m}$, achieving

$$\|\boldsymbol{R}+\boldsymbol{Q}\|_\infty \le \alpha \tag{B.3}$$

can be written

$$\boldsymbol{Q}=\mathfrak{F}_U[\Phi] \tag{B.4}$$

where Φ is an arbitrary transfer function constrained to satisfy $\Phi \in RH_\infty^{p\times m}$, $\|\Phi\|_\infty \le 1$, and

$$U=\left[\begin{array}{cc|c} U_{11} & -\alpha U_{12} & U_{12} \\ \hline U_{21} & I-\alpha U_{22} & U_{22}\end{array}\right]. \tag{B.5}$$

With state-space form

$$U=\left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22}\end{array}\right] = \left[\begin{array}{c|c:c} -A^* & W^{*-1}C^*D_\perp & \alpha^{-1}W^{*-1}QB \\ \hline -(CP+DB^*) & D_\perp & -\alpha^{-1}D \\ \hdashline B^* & 0 & \alpha^{-1}I \end{array}\right] \tag{B.6}$$

and $-Q$ (respectively, $-P$) is the controllability (respectively, observability) Gramian of $\boldsymbol{R}^*$, $W \triangleq (PQ - \alpha^2 I)$, and $D_\perp$ is the unitary completion of D, i.e., $[D_\perp, D]$ is square unitary.

Proof: Noting that in Lemma B.1 that ϕ is an *arbitrary* contraction, V_{11}, V_{21} can be postmultiplied by a unitary matrix without changing the parametrization in (B.1). Next note that $\boldsymbol{R}^*\boldsymbol{R} = I \Rightarrow D^*D = I$. Hence, if the unitary completion of D, $D_\perp$ is chosen, so that the matrix

$$S=[D_\perp, D] \tag{B.7}$$

is unitary, i.e., $SS^* = S^*S = I$, then a matrix U, where

$$U = \left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22} \end{array}\right] = \left[\begin{array}{c|c} V_{11}S & V_{12} \\ \hline V_{21}S & V_{22} \end{array}\right]$$

will also parametrize all Q in (B.1) and $\Phi = S^{-1}\phi$ again satisfies $\|\Phi\|_\infty \le 1$. Note from (B.2) that

$$\begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} S$$

$$= \left[\begin{array}{c|cc} -A^* & W^{*-1}C^*D_\perp & W^{*-1}C^*D \\ \hline -(CP+DB^*) & D_\perp & D \\ B^* & 0 & 0 \end{array}\right]. \tag{B.8}$$

Further, R^* has a state-space realization given by $(-A^*, C^*, -B^*, D^*)$, and noting from [12], [5] that for a coinner function $C^*D = -QB$, then the result in (B.6) is immediate. ■

Appendix C

Proof of Lemma 4.1: The proof is straightforward by noting the following lemma due to Redheffer.

Lemma C.1 (Redheffer [28]): For $J, K \in RL_\infty$ with $\|J\|_\infty \le \sigma$, $\|J_{22}K\|_\infty < 1$, then $\|\mathfrak{F}_L(J, K)\|_\infty \le \sigma$ if $\|K\|_\infty \le \sigma^{-1}$. ■

Proof of Lemma 4.1: First, to prove a), let F be defined as

$$F = \mathfrak{F}_L\left(J, \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}\right)$$

where

$$J = \left[\begin{array}{c|cc} 0 & \alpha^{-1}I & 0 \\ \hline \sqrt{\alpha^{-2}-1}\,I & 0 & I \end{array}\right].$$

Noting that as $JJ^* = \alpha^{-2}I$, then

$$\|F\| \le \alpha^{-1} \text{ (by Lemma C.1).}$$

Now by (2.1a)

$$F = \alpha^{-1}\sqrt{\alpha^{-2}-1}\,E_1(I-E_2)^{-1}$$

$$\Rightarrow \|E_1(I-E_2)^{-1}\|_\infty \le \alpha(1-\alpha^2)^{-1/2}.$$

To prove b), note that the selection $E_1 = (1-\alpha^2)F$, $E_2 = \alpha I$ demonstrates that for any $F \in RH_\infty$ satisfying $\|F\|_\infty \le \alpha(1-\alpha^2)^{-1/2}$ there exists $E_1, E_2 \in RH_\infty$ such that $E_1(I-E_2)^{-1} = F$ satisfying

$$\left\| \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \right\|_\infty \le \alpha.$$

Proof of Lemma 4.2: The proof of this lemma uses a well-known result from the Hankel operator theory.

Lemma C.2 (Francis [8, p. 70]): Let

$$\|R - X\|_\infty = \|R^*\|_H \triangleq \sigma_1(R^*) \tag{C.1}$$

where $R^*, X \in RH_\infty$. Then, there exist vectors $g(s)$ and $f(s) \in RH_2$ independent of X such that

$$(R - X)g(s) = \sigma_1(R^*)f(-s). \tag{C.2}$$

■

The proof of Lemma 4.2 is then as follows.

i) It is well known that $\|[\tilde{N}, \tilde{M}]\|_H \le \|[\tilde{N}, \tilde{M}]\|_\infty = 1$.

ii) Suppose $\|[\tilde{N}, \tilde{M}]\|_H = 1$, then from Lemma C.2, there exists $g(s), f(s) \in RH_2$ such that

$$\begin{bmatrix} \tilde{N}^* \\ \tilde{M}^* \end{bmatrix} g(s) = f(-s) \tag{C.3}$$

(i.e., $R = \{ {}^{-\tilde{N}^*}_{\tilde{M}^*}]$, and $X = 0$ in Lemma C.2).

But, one of the requirements for a coprime factorization (see Definition 2.1) is that there exist $U, V \in RH_\infty$

$$U^*\tilde{N}^* + V^*\tilde{M}^* = I. \tag{C.4}$$

Premultiplying (C.3) by $[U^*, V^*]$ yields

$$g(s) = [U^*, V^*]f(-s)$$

which is a contradiction as the right-hand side $\notin RH_2$. ■

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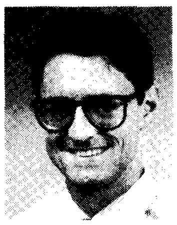
References

[1] B. Anderson and Y. Liu, "Controller reduction: Concepts and approaches," in *Proc. Acc*, MN, 1987, pp. 1-9.
[2] R. Bucy, "The Riccati equation and its bounds," *J. Comput. Syst. Sci.*, vol. 6, pp. 343-353, 1972.
[3] M. Chen and C. Desoer, "Necessary and sufficient for robust stability of linear distributed feedback systems," *Int. J. Contr.*, vol. 35, pp. 255-267, 1982.
[4] C. Chu, J. Doyle, and B. Lee, "The general distance problem in H_∞ optimal control theory," *Int. J. Contr.*, vol. 44, pp. 565-596, 1986.
[5] J. Doyle, *Lecture Notes for ONR/Honeywell Workshop on Advances in Multivariable Control*, Minneapolis, MN, 1984.
[6] J. Doyle and G. Stein, "Multivariable feedback design: Concepts for a classical/modern synthesis," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 4-16, 1981.
[7] Y. Foo and I. Postlethwaite, "An H_∞-minimax approach to the design of robust control systems," *Syst. Contr. Lett.*, vol. 5, pp. 81-88, 1984.
[8] B. A. Francis, *A Course in H_∞ Control Theory*. New York: Springer-Verlag, 1987.
[9] K. Glover and J. Doyle, "State-space formulae for all stabilizing controllers that satisfy an H_∞ norm bound and relations to risk sensitivity," *Syst. Contr. Lett.*, vol. 11, pp. 167-172, 1988.
[10] K. Glover, "Robust stabilization of linear multivariable systems: Relations to approximation," *Int. J. Contr.*, vol. 43, pp. 741-766, 1986.
[11] ——, "Model reduction: A tutorial on Hankel norm methods and lower bounds on L_2 errors," in *Proc. 10th IFAC Congress*, Vol. X, Munich, Germany, Pergamon, 1987, pp. 288-293.
[12] ——, "All optimal Hankel norm approximations of linear multivariable systems and their L_∞-error bounds," *Int. J. Contr.*, vol. 39, pp. 1115-1193, 1984.
[13] K. Glover and D. McFarlane, "Robust stabilization of normalized coprime factors: An explicit H_∞ solution," in *Proc. Amer. Contr. Conf.*, Atlanta, GA, 1988, pp. 842-847, 1988.
[14] M. Grimble, "Optimal H_∞ multivariable robust controllers and the relationship to LQG design problems," *Int. J. Contr.*, vol. 48, no. 1, pp. 35-58, 1988.
[15] E. Jonckheere and L. Silverman, "A new set of invariants for linear systems—Applications to reduced order compensator design," *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 953-964, 1983.
[16] R. Kalman, "A new approach to linear filtering and prediction problems," *J. Basic Eng.*, pp. 35-45, Mar. 1960.
[17] P. Khargonekar, T. Georgiou, and A. Pascoal, "On the robust stabilizability of linear time-invariant plants with unstructured uncertainty," *IEEE Trans. Automat. Contr.*, vol. AC-32, no. 3, pp. 201-207, 1987.
[18] H. Kimura, "Robust stabilization for a class of transfer functions," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 788-793, 1984.
[19] D. Limebeer and G. Halikias, "A controller degree bound for H_∞-optimal controllers of the second kind," *SIAM J. Contr. Opt.*, vol. 26, pp. 646-677, 1988.
[20] Y. Liu and B. Anderson, "Controller reduction via stable factorization and balancing," *Int. J. Contr.*, pp. 507-531, 1986.

[21] D. McFarlane, K. Glover, and M. Noton, "Robust stabilization of a flexible space-platform: An H_∞ coprime factorization approach," in *Proc. Contr. 88' IEE Conf.*, Oxford, England, 1988.
[22] D. McFarlane and K. Glover, "An H_∞ design procedure using robust stabilization of normalized coprime factors," in *Proc. 1988 Conf. Decision Contr.*, Houston, TX, 1988, pp. 1343-1348.
[23] D. McFarlane, "Robust controller design using normalized coprime factor plant descriptions," Ph.D. dissertation, Univ. Cambridge, Cambridge, England, 1988.
[24] D. McFarlane, K. Glover, and M. Vidyasagar, "Reduced order controller design using coprime factor model reduction," *IEEE Trans. Automat. Contr.*, to be published.
[25] D. Meyer and G. Franklin, "A connection between normalized coprime factorizations and linear quadratic regulator theory," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 227-228, 1987.
[26] D. Meyer, "Model reduction via fractional representations," Ph.D. dissertation, Stanford Univ., Stanford, CA, 1987.
[27] D. Mustafa and K. Glover, "Controllers which satisfy a closed-loop H_∞ norm bound and maximize an entropy integral," in *Proc. 1988 Conf. Decision Contr.*, Houston, TX, pp. 959-964.
[28] M. Redheffer, "On a certain linear fractional transformation," *J. Math. Phys.*, vol. 39, pp. 269-286, 1960.
[29] M. Safanov, R. Chiang, and D. Limebeer, "Hankel model reduction without balancing—A descriptor approach," in *Proc. 1987 Conf. Decision Contr.*, 1987.
[30] M. Verma, J. Helton, and E. Jonckheere, "Robust stabilization of a family of plants with varying numbers of right half-plane poles," in *Proc. Automat. Contr. Conf.*, Seattle, WA, 1986, pp. 1827-1832.
[31] M. Vidyasagar, H. Schneider, and B. Francis, "Algebraic and topological aspects of feedback stabilization," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 880-894, 1982.
[32] M. Vidyasagar, "The graph metric for unstable plants and robustness estimates for feedback stability," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 403-417, 1984.
[33] ——, *Control System Synthesis: A Coprime Factorization Approach*. Cambridge, MA: M.I.T. Press, 1985.
[34] ——, "Normalized coprime factorizations for nonstrictly proper systems," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 300-301, 1988.
[35] M. Vidyasagar and H. Kimura, "Robust controllers for uncertain linear multivariable systems," *Automatica*, pp. 85-94, 1986.
[36] J. Willems, *The Analysis of Feedback Systems*. Cambridge, MA: M.I.T. Press, 1971.
[37] G. Zames, "Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms and approximate inverses," *IEEE Trans. Automat. Contr.*, vol. AC-26, 1981.

Keith Glover (S'71–M'73) was born in Bromley, Kent, England, in 1946. He received the B.Sc.(Eng) degree from Imperial College, London, England, in 1967, and the S.M., E.E., and Ph.D. degrees from the Massachusetts Institute of Technology, Cambridge, in 1971, 1971, and 1973, respectively, all in electrical engineering.

From 1967 to 1969 he was a Development Engineer with the Marconi Company. From 1973 to 1976 he was on the faculty of the University of Southern California, Los Angeles. Since 1976 he has been with the Department of Engineering, University of Cambridge, Cambridge, England, where he is currently a Reader. He was a Visiting Fellow at the Australian National University, Canberra, from 1983 to 1984. His current research interests include linear systems, model approximation, robust control, and identification.



Duncan McFarlane (S'87–M'87–S'88–M'88) was born in Melbourne, Australia, in 1963. He received the B. Eng. degree in mechanical engineering from the University of Melbourne, Melbourne, Australia, in 1984, and the Ph.D. degree in control from the University of Cambridge, Cambridge, England, in 1988.

He has been employed by the Broken Hill Proprietary Co., Ltd., Clayton, Australia, since 1980, and is currently a Research Officer in the BHP Melbourne Research Laboratories. His research interests include robust control, optimal control, and modeling and control of steel processes.